

## ANALYTIC FUNCTIONAL CALCULUS FOR TWO OPERATORS

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ABSTRACT. Properties of the mappings

$$C \mapsto \frac{1}{(2\pi i)^2} \int_{\Gamma_1} \int_{\Gamma_2} f(\lambda, \mu) R_{1,\lambda} C R_{2,\mu} d\mu d\lambda,$$

$$C \mapsto \frac{1}{2\pi i} \int_{\Gamma} g(\lambda) R_{1,\lambda} C R_{2,\lambda} d\lambda$$

are discussed; here  $R_{1,(\cdot)}$  and  $R_{2,(\cdot)}$  are pseudo-resolvents, i. e., resolvents of bounded, unbounded, or multivalued linear operators, and  $f$  and  $g$  are analytic functions. Several applications are considered: a representation of the impulse response of a second order linear differential equation with operator coefficients, a representation of the solution of the Sylvester equation, and an exploration of properties of the differential of the ordinary functional calculus.

## 1. INTRODUCTION

Let  $A$  and  $B$  be matrices of the sizes  $n \times n$  and  $m \times m$  respectively. The result  $p(A, B)$  of the substitution of these matrices into a polynomial  $p(\lambda, \mu) = \sum_{i,j=0}^N c_{ij} \lambda^i \mu^j$  of two variables is usually understood to be the mapping  $C \mapsto \sum_{i,j=0}^N c_{ij} A^i C B^j$  acting on  $n \times m$ -matrices, or to be the block matrix  $\{a_{ij} B\}$ . This article is devoted to extensions and applications of this construction.

First, the matrices  $A$  and  $B$  can be replaced by bounded linear operators acting in (infinite-dimensional) Banach spaces  $X$  and  $Y$  respectively. In this case, the number of interpretations of the object  $p(A, B)$  increases. The most natural abstract interpretation is considering  $p(A, B)$  as an operator acting in the completion of the algebraic tensor product [28, 64, 112]  $X \otimes Y$  with respect to some cross-norm. This interpretation embraces many spaces of functions of two variables. For example [28, 64, 112],  $L_1[a, b] \overline{\otimes}_{\pi} L_1[c, d]$  is isometrically isomorphic to  $L_1[a, b] \times [c, d]$ , and  $C[a, b] \overline{\otimes}_{\varepsilon} C[c, d]$  is isometrically isomorphic to  $C[a, b] \times [c, d]$ . But unfortunately,  $L_{\infty}[a, b] \overline{\otimes}_{\varepsilon} L_{\infty}[c, d]$  is isomorphic only to a subspace of the space  $L_{\infty}[a, b] \times [c, d]$ . Another example, which can not be treated directly in terms of tensor products, is the interpretation of  $p(A, B)$  as the transformation  $C \mapsto \sum_{i,j=0}^N c_{ij} A^i C B^j$  of the operators  $C : Y \rightarrow X$ . From the point of view of applications, the last example seems to be the most important. Therefore, in all cases, we call the operator that corresponds to  $p(A, B)$  a *transformator*; this term is conventional [51] for the mappings of the type  $C \mapsto \sum_{i,j=0}^N c_{ij} A^i C B^j$  acting on operators  $C$ . In order to embrace the last example and some others, our treatment is based on the notion of an extended tensor product (Section 5) proposed in [89].

Second, one may replace the polynomial  $p$  by an analytic function  $f$ . In this case, it is convenient to define  $f(A, B)$  by means of a contour integral. For our main interpretation of  $f(A, B)$  as a transformator acting on operators  $C : Y \rightarrow X$ , the relevant formula looks

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as follows:

$$f(A, B)C = \frac{1}{(2\pi i)^2} \int_{\Gamma_1} \int_{\Gamma_2} f(\lambda, \mu)(\lambda \mathbf{1} - A)^{-1} C (\mu \mathbf{1} - B)^{-1} d\mu d\lambda. \quad (*)$$

We call a correspondence of the type  $f \mapsto f(A, B)$  that maps functions  $f$  to transformers (operators)  $f(A, B)$  a *functional calculus*.

From the algebraic point of view, the functional calculus  $f \mapsto f(A, B)$  possesses properties of the tensor product  $\varphi_1 \otimes \varphi_2$  of two ordinary functional calculi

$$\varphi_1(f) = \frac{1}{2\pi i} \int_{\Gamma_1} f(\lambda)(\lambda \mathbf{1} - A)^{-1} d\lambda, \quad \varphi_2(f) = \frac{1}{2\pi i} \int_{\Gamma_2} f(\mu)(\mu \mathbf{1} - B)^{-1} d\mu.$$

To emphasise this fact, we use the notation  $\varphi_1 \boxtimes \varphi_2$  and write  $[(\varphi_1 \boxtimes \varphi_2)f]C$  instead of  $f(A, B)C$  when the transformer  $f(A, B)$  acts in an extended tensor product. An important and nontrivial property of the transformation  $\varphi_1 \boxtimes \varphi_2$  is the spectral mapping theorem (Theorem 39).

Third, the basic properties of the functional calculus  $\varphi_1 \boxtimes \varphi_2$  are preserved when one replaces the resolvents  $R_\lambda = (\lambda \mathbf{1} - A)^{-1}$  of operators by *pseudo-resolvents* [67], i. e. operator-valued functions  $R_{(\cdot)}$  that satisfy the Hilbert identity

$$R_\lambda - R_\mu = -(\lambda - \mu)R_\lambda R_\mu.$$

This generalization enables one to cover some additional examples. For example, a special case of a pseudo-resolvent is [67] the resolvent of an unbounded operator, and the most general example of a pseudo-resolvent is the resolvent of a linear relation or, in other terminology, a multivalued linear operator [8, 9, 16, 20, 38, 57, 92].

Moreover, in this article, we adhere to the point of view that a pseudo-resolvent is as a fundamental object as an operator (bounded, unbounded or multivalued) that generates it. The reason for this stems from the fact that when speaking of unbounded operators and linear relations we often actually work with their resolvents. For example, an unbounded operator is a generator of a strongly continuous or analytic semigroup if and only if [36, 67, 127] its resolvent satisfies a special estimate of the decay rate at infinity; in [75, Theorem 2.25] and [106, VIII.7], the natural convergence of unbounded operators is defined as the convergence of their resolvents in norm; and in [105, 107], a function  $f$  of unbounded operators  $A$  and  $B$  is defined as an (unbounded) operator  $f(A, B)$  that possesses the following property: there exist sequences of bounded operators  $A_n$  and  $B_n$  such that the resolvents of  $A_n$ ,  $B_n$ , and  $f(A_n, B_n)$  converge in norm to the resolvents of  $A$ ,  $B$ , and  $f(A, B)$  respectively. Another argument (not used in this article) is that there is no analogue of unbounded and multivalued operators in Banach algebras, but, nevertheless, there are evident analogues of the resolvents of such operators.

This approach enables one to extend the notion of  $f(A, B)$  to meromorphic functions  $f$  (Theorem 42): we define the result of the action of a meromorphic function on  $A$  and  $B$  to be a new pseudo-resolvent and do not discuss which operator it is generated by. Along the way, we answer (Corollary 43) the question of the independence of the definition of  $f(A, B)$  for unbounded  $A$  and  $B$  posed in [105, 107] (see the previous paragraph) from the choice of approximating sequences  $A_n$  and  $B_n$ .

Many important applications are connected with the special cases of construction (\*) and their modifications. For example, it often occurs that the function  $f$  depends on the difference or the sum of its arguments: the transformer  $C \mapsto AC - CB$  generated by the function  $f(\lambda, \mu) = \lambda - \mu$  is related to the Sylvester equation (Section 10), and the transformer  $C \mapsto e^{At} C e^{Bt}$  generated by the function  $f(\lambda, \mu) = e^{t(\lambda + \mu)}$  is connected with the stability theory of differential equations [5, 23].

The version (Section 8)

$$f^{[1]}(A, B)C = \frac{1}{2\pi i} \int_{\Gamma} f(\lambda)(\lambda \mathbf{1} - A)^{-1} C (\lambda \mathbf{1} - B)^{-1} d\lambda \quad (**)$$

of functional calculus (\*) frequently occurs in applications; it involves functions  $f$  of one variable. For example, expression (\*\*) with  $f(\lambda) = e^{\lambda t}$  forms the principal part of the representation of a solution of the second order differential equation  $(\frac{d}{dt} - A)C(\frac{d}{dt} - B)y = 0$  (Section 9). Further (Section 11), the differential of the ordinary functional calculus  $A \mapsto f(A)$  at a point  $A$  can be represented in the form (Theorem 67)

$$df(\cdot, A) = f^{[1]}(A, A).$$

It turns out that (Theorems 45)  $f^{[1]}(A, B)C$  coincides with (\*) provided  $f^{[1]}$  is understood to be the divided difference

$$f^{[1]}(\lambda, \mu) = \begin{cases} \frac{f(\lambda) - f(\mu)}{\lambda - \mu}, & \text{if } \lambda \neq \mu, \\ f'(\lambda), & \text{if } \lambda = \mu, \\ 0, & \text{if } \lambda = \infty \text{ or } \mu = \infty. \end{cases}$$

When choosing the level of generality of our exposition, we proceed from the following principles. First, in order that a specialist in the classical operator theory can use the article we are trying to minimize explicit mention of Banach algebras and linear relations (multivalued operators) at least in main statements. At the same time, when using the operator language, we aspire the maximal generality and, in particular, where possible, consider the case of an arbitrary pseudo-resolvent (and thereby, implicitly, the cases of unbounded operators and linear relations). Second, we are trying not to fall outside the framework of the theory of analytic functions of an operator and thus, for example, do not discuss issues related to the generators of semigroups. Third, as far as possible, we avoid the implicit use of operator pencils  $\lambda \mapsto \lambda F - G$  with  $F \neq \mathbf{1}$  because this approach leads to very cumbersome formulae. Finally, we restrict ourselves to the consideration of functions of two variables, suggesting that the generalization to three and more variables will not cause significant difficulties.

The literature on the subject under discussion is extremely extensive. Therefore, the bibliography can not be made comprehensive; the presented references reflect authors' tastes and interests. Many additional references can be found in the cited articles and books.

Sections 2–5 outlines preliminary information. Here we recall and fix notation and the main facts in a convenient form. In Section 2, the terminology connected with Banach algebras and their properties is recalled. In Section 3, the basic properties of algebras of analytic functions of one and two variables are described. In Section 4, we discuss the notion of pseudo-resolvent and recall the construction of the functional calculus of analytic functions of one variable (Theorems 25 and 26) including the spectral mapping theorem (Theorem 27). In Section 5, the definition of the extended tensor product is given, the main examples are described, and the construction of the functional calculus of operator-valued analytic functions of one variable (Theorems 28 and 29) is recalled as well as the relevant spectral mapping theorem (Theorem 31).

In Section 6, we present the construction of the functional calculus (\*) of functions of two variables (Theorems 32, 33, and 34) and prove the corresponding spectral mapping theorem (Theorem 39). In Section 7, we extend these results to meromorphic functions. A well-known example of a meromorphic function of an operator is a polynomial of an unbounded operator (a polynomial has a pole at infinity, and the point at infinity belongs

to the extended spectrum of an unbounded operator). This example shows that the result of applying a meromorphic function cannot be a bounded transformator; as a convenient tool for its description we use not the resulting object itself, but its resolvent, and we interpret the extended singular set of this resolvent as its spectrum (Theorem 42).

In Section 8, we discuss modified variant (\*\*) of the functional calculus of functions of two variables. The connection between functional calculi (\*) and (\*\*) is established as well as some properties of functional calculus (\*\*). The subsequent sections are devoted to applications. In Section 9, the pencil  $\lambda \mapsto \lambda^2 E + \lambda F + H$  of the second order is considered; it is induced by the equation  $E\ddot{y}(t) + F\dot{y}(t) + Hy(t) = 0$ ; we assume that the pencil admits a factorization, i. e. the representation in the form of a product of two linear pencils. In such a case, the solution of the differential equation is expressed by a transformation of the kind (\*\*) (Theorem 54). In Section 10, we discuss the properties of the transformator  $Q : C \mapsto Z$  generated by the Sylvester equation  $AZ - ZB = C$  (Theorem 62). Finally, in Section 11, it is shown that the differential of the ordinary functional calculus  $A \mapsto f(A)$  is also a kind of transformator (\*\*) (Theorem 67).

## 2. BANACH ALGEBRAS

In this Section, we clarify the terminology connected with Banach algebras [18, 67, 110] and recall some of their properties.

In this article, all linear spaces and algebras are assumed to be complex.

Let  $X$  and  $Y$  be Banach spaces. We denote by  $\mathbf{B}(X, Y)$  the set of all bounded linear operators  $A : X \rightarrow Y$ . When  $X = Y$ , we use the shorthand symbol  $\mathbf{B}(X)$ . The symbol  $\mathbf{1} = \mathbf{1}_X$  stands for the identity operator. We adhere to the following notations:  $X^*$  denotes the conjugate space of  $X$ ;  $\langle x, x^* \rangle$  denotes the value of the functional  $x^* \in X^*$  on  $x \in X$ , and  $\langle x^{**}, x^* \rangle$  is the value of  $x^{**} \in X^{**}$  on  $x^* \in X^*$ ;  $A^*$  denotes the conjugate operator of  $A \in \mathbf{B}(X, Y)$ . The *preconjugate* of an operator  $A \in \mathbf{B}(Y^*, X^*)$  is an operator  $A^0 \in \mathbf{B}(X, Y)$  such that  $(A^0)^* = A$ .

The *unit* [18, 67, 110] of an algebra  $\mathbf{B}$  is an element  $\mathbf{1} \in \mathbf{B}$  such that  $\mathbf{1}A = A\mathbf{1}$  for all  $A \in \mathbf{B}$ . If an algebra  $\mathbf{B}$  has a unit, it is called *an algebra with a unit* or *unital*.

A subset  $\mathbf{R}$  of an algebra  $\mathbf{B}$  is called a *subalgebra* if  $\mathbf{R}$  is stable under the algebraic operations (addition, scalar multiplication, and multiplication), i. e.  $A + B, \lambda A, AB \in \mathbf{R}$  for all  $A, B \in \mathbf{R}$  and  $\lambda \in \mathbb{C}$ . If the unit  $\mathbf{1}$  of an algebra  $\mathbf{B}$  belongs to its subalgebra  $\mathbf{R}$ , then  $\mathbf{R}$  is called a *subalgebra with a unit* or a *unital subalgebra*.

Let  $\mathbf{B}$  be a non-unital algebra. The set  $\tilde{\mathbf{B}} = \mathbb{C} \oplus \mathbf{B}$  with the componentwise linear operations and the multiplication  $(\alpha, A)(\beta, B) = (\alpha\beta, \alpha B + \beta A + AB)$  is obviously an algebra with the unit  $(1, \mathbf{0})$ , where  $\mathbf{0}$  is the zero of the algebra  $\mathbf{B}$ . The algebra  $\tilde{\mathbf{B}}$  is called the algebra derived from  $\mathbf{B}$  by adjoining a unit or an algebra with an *adjoint unit*. The symbol  $\mathbf{1}$  stands for the element  $(1, \mathbf{0})$ , and the symbol  $\alpha\mathbf{1} + A$  denotes the element  $(\alpha, A)$ . If  $\mathbf{B}$  is a normed algebra, we set  $\|(\alpha, A)\| = |\alpha| + \|A\|$ . Clearly, this formula defines a norm on  $\tilde{\mathbf{B}}$ . It is also clear that  $\tilde{\mathbf{B}}$  is complete provided that  $\mathbf{B}$  is complete. If  $\mathbf{B}$  is unital, then  $\tilde{\mathbf{B}}$  means the algebra  $\mathbf{B}$  itself.

**Theorem 1.** *Let  $\mathbf{B}$  be a unital Banach algebra and  $A, B \in \mathbf{B}$ . If  $A$  is invertible and*

$$\|B\| \cdot \|A^{-1}\| < 1,$$

*then the element  $A - B$  is also invertible and*

$$(A - B)^{-1} = A^{-1} + A^{-1}BA^{-1} + A^{-1}BA^{-1}BA^{-1} + A^{-1}BA^{-1}BA^{-1}BA^{-1} + \dots$$

In this case

$$\begin{aligned}
\|(A - B)^{-1}\| &\leq \frac{\|A^{-1}\|}{1 - \|B\| \cdot \|A^{-1}\|}, \\
\|(A - B)^{-1} - A^{-1}\| &\leq \frac{\|B\| \cdot \|A^{-1}\|^2}{1 - \|B\| \cdot \|A^{-1}\|}, \\
\|(A - B)^{-1} - A^{-1} - A^{-1}BA^{-1}\| &\leq \frac{\|B\|^2 \cdot \|A^{-1}\|^3}{1 - \|B\| \cdot \|A^{-1}\|}.
\end{aligned} \tag{1}$$

*Proof.* We consider the series

$$A^{-1} + A^{-1}BA^{-1} + A^{-1}BA^{-1}BA^{-1} + A^{-1}BA^{-1}BA^{-1}BA^{-1} + \dots$$

We represent this series in the form  $A^{-1}(\mathbf{1} + BA^{-1} + BA^{-1}BA^{-1} + BA^{-1}BA^{-1}BA^{-1} + \dots)$ . Since  $\|BA^{-1}\| \leq \|B\| \cdot \|A^{-1}\| < 1$ , the series converges absolutely. We denote its sum by  $C$ . It is straightforward to verify that  $C$  coincides with the inverse of  $A - B$ .

Estimates (1) follow from the geometric series formula  $\sum_{k=0}^{\infty} q^k = \frac{1}{1-q}$ ,  $|q| < 1$ . For example, let us prove the second estimate:

$$\|(A - B)^{-1} - A^{-1}\| \leq \sum_{k=1}^{\infty} \|B\|^k \cdot \|A^{-1}\|^{k+1} = \frac{\|B\| \cdot \|A^{-1}\|^2}{1 - \|B\| \cdot \|A^{-1}\|}. \quad \square$$

Let  $\mathbf{B}$  be a (nonzero) unital algebra and  $A \in \mathbf{B}$ . The set of all  $\lambda \in \mathbb{C}$  such that the element  $\lambda\mathbf{1} - A$  is not invertible is called the *spectrum* of the element  $A$  (in the algebra  $\mathbf{B}$ ) and is denoted by the symbol  $\sigma(A)$ . The complement  $\rho(A) = \mathbb{C} \setminus \sigma(A)$  is called the *resolvent set* of  $A$ . The function  $R_\lambda = (\lambda\mathbf{1} - A)^{-1}$  is called the *resolvent* of the element  $A$ .

**Proposition 2** ([67, Theorem 4.8.1]). *The resolvent  $R_{(\cdot)}$  of any element  $A \in \mathbf{B}$  satisfies the Hilbert identity*

$$R_\lambda - R_\mu = -(\lambda - \mu)R_\lambda R_\mu, \quad \lambda, \mu \in \rho(A). \tag{2}$$

**Proposition 3** ([18, ch.1, § 2, . 5], [110, Theorem 10.13]). *The spectrum of any element  $A$  of a nonzero unital Banach algebra  $\mathbf{B}$  is a compact and nonempty subset of the complex plane  $\mathbb{C}$ .*

Let  $\mathbf{A}$  and  $\mathbf{B}$  be algebras. A mapping  $\varphi: \mathbf{A} \rightarrow \mathbf{B}$  is called [18] a *morphism of algebras* if

$$\begin{aligned}
\varphi(A + B) &= \varphi(A) + \varphi(B), \\
\varphi(\alpha A) &= \alpha\varphi(A), \\
\varphi(AB) &= \varphi(A)\varphi(B).
\end{aligned}$$

If, in addition,  $\mathbf{A}$  and  $\mathbf{B}$  are unital and

$$\varphi(\mathbf{1}_\mathbf{A}) = \mathbf{1}_\mathbf{B},$$

$\varphi$  is called a *morphism of unital algebras*. If  $\mathbf{A}$  and  $\mathbf{B}$  are Banach algebras [18, 67, 110] and, in addition, the morphism  $\varphi$  is continuous, then  $\varphi$  is called a *morphism of Banach algebras*.

A unital subalgebra  $\mathbf{R}$  of a unital algebra  $\mathbf{B}$  is called [18, ch. 1, § 1.4] *full* if every  $B \in \mathbf{R}$  which is invertible in  $\mathbf{B}$  is also invertible in  $\mathbf{R}$ . Since the inverse is unique, the last definition is equivalent to the following: if for  $B \in \mathbf{R}$  there exists  $B^{-1} \in \mathbf{B}$  such that  $BB^{-1} = B^{-1}B = \mathbf{1}$ , then  $B^{-1} \in \mathbf{R}$ .

*Example 1.* Let  $X$  be a Banach space. The set  $\mathbf{B}_0(X^*)$  of all operators that have a preconjugate is a full subalgebra of the algebra  $\mathbf{B}(X^*)$ .

**Proposition 4** ([18, ch. 1, § 2.5]). *The closure of a full subalgebra of a Banach algebra is also a full subalgebra. The closure of the least full subalgebra of a Banach algebra that contains a set  $M$  is the least full closed subalgebra that contains  $M$ .*

An algebra  $\mathbf{B}$  is called *commutative* if  $AB = BA$  for all  $A, B \in \mathbf{B}$ .

A *character* of a unital commutative algebra  $\mathbf{B}$  [18, ch. 1, § 1.5] is a morphism  $\chi: \mathbf{B} \rightarrow \mathbb{C}$  of unital algebras. A *character* of a commutative non-unital algebra  $\mathbf{B}$  [18, ch. 1, § 1.5] is a morphism of (non-unital) algebras  $\chi: \mathbf{B} \rightarrow \mathbb{C}$ . If an algebra  $\mathbf{B}$  is non-unital, we denote by  $\chi_0$  the character  $\chi_0: \mathbf{B} \rightarrow \mathbb{C}$  that is equal to zero on all elements of  $\mathbf{B}$ . We call  $\chi_0$  the *zero character*. We stress that the character  $\chi_0$  exists only if the algebra  $\mathbf{B}$  is non-unital.

**Proposition 5.** *All characters of a non-unital algebra  $\mathbf{B}$  are extendable uniquely to characters of the algebra  $\tilde{\mathbf{B}}$  derived from  $\mathbf{B}$  by adjoining a unit. The extension is defined by the formula  $\chi(\alpha\mathbf{1} + A) = \alpha + \chi(A)$ . Conversely, the restriction of any character of the algebra  $\tilde{\mathbf{B}}$  to  $\mathbf{B}$  is a character of the algebra  $\mathbf{B}$ . In particular, the zero character  $\chi_0$  is the restriction of the character  $\alpha\mathbf{1} + A \mapsto \alpha$ ; we will denote it by the same symbol  $\chi_0$ .*

*Proof.* The proof is obvious. □

We denote by  $X(\mathbf{B})$  the set of all *nonzero* characters of a commutative algebra  $\mathbf{B}$  (unital or non-unital), and we denote by  $\tilde{X}(\mathbf{B})$  the set of all characters of a commutative algebra  $\mathbf{B}$  (including the zero character  $\chi_0$  if the algebra is non-unital). If an algebra  $\mathbf{B}$  is unital, then  $\tilde{X}(\mathbf{B})$  obviously coincides with  $X(\mathbf{B})$ . The set  $X(\mathbf{B})$  is called [18] the *character space* of the algebra  $\mathbf{B}$ .

**Theorem 6** ([18, ch. 1, § 3.3, Proposition 3]). *Let  $\mathbf{B}$  be a unital commutative Banach algebra. Then for all  $A \in \mathbf{B}$ ,*

$$\sigma(A) = \{ \chi(A) : \chi \in \tilde{X}(\mathbf{B}) \}.$$

**Corollary 7** ([18, ch. 1, § 3, Theorem 1]). *Every character of a commutative Banach algebra is continuous; namely, its norm is less than or equal to unity.*

**Corollary 8.** *In a unital commutative Banach algebra  $\mathbf{B}$ , the spectrum continuously depends on an element; more precisely, if  $A, B \in \mathbf{B}$  and  $\|A - B\| < \varepsilon$ , then  $\sigma(B)$  is contained in the  $\varepsilon$ -neighbourhood of  $\sigma(A)$ .*

### 3. ALGEBRAS OF ANALYTIC FUNCTIONS

This Section is a preparation for a discussion of analytic functional calculi. Here we collect some preliminaries on algebras of analytic functions defined on subsets of  $\overline{\mathbb{C}}$  and  $\overline{\mathbb{C}}^2$ .

We denote by  $\overline{\mathbb{C}}$  the one-point compactification  $\mathbb{C} \cup \{\infty\}$  of the complex plane  $\mathbb{C}$ , and we denote by  $\overline{\mathbb{C}}^2$  the Cartesian product  $\overline{\mathbb{C}} \times \overline{\mathbb{C}}$ .

**Proposition 9.** *Let  $\sigma_1, \sigma_2 \subseteq \overline{\mathbb{C}}$  be closed sets, and let an open set  $W \subseteq \overline{\mathbb{C}}^2$  contains  $\sigma_1 \times \sigma_2$ . Then there exist open sets  $U, V \subseteq \overline{\mathbb{C}}$  such that  $\sigma_1 \times \sigma_2 \subseteq U \times V \subseteq W$ .*

*Proof.* For an arbitrary  $\lambda \in \sigma_1$ , we consider the set  $\{\lambda\} \times \sigma_2$ . We consider a finite cover of  $\{\lambda\} \times \sigma_2$  by the sets of the form  $U_i \times V_i$ , where  $U_i, V_i \subseteq \overline{\mathbb{C}}$  are open and  $U_i \times V_i \subseteq W$ . We put  $\tilde{U} = \bigcap_i U_i$  and  $\tilde{V} = \bigcup_i V_i$ . It is clear that the set  $\tilde{U} \times \tilde{V} \subseteq W$  also covers the set  $\{\lambda\} \times \sigma_2$ , namely,  $\{\lambda\} \subseteq \tilde{U}$ ,  $\sigma_2 \subseteq \tilde{V}$ .

Further, we cover every subset of the form  $\{\lambda\} \times \sigma_2$  of the set  $\sigma_1 \times \sigma_2$  by a set of the form  $\tilde{U} \times \tilde{V} \subseteq W$ . We choose a finite subcover  $\{\tilde{U}_k \times \tilde{V}_k\}$  and put  $U = \cup_k \tilde{U}_k$ ,  $V = \cap_k \tilde{V}_k$ . Obviously,  $\sigma_1 \times \sigma_2 \subseteq U \times V \subseteq W$ .  $\square$

**Proposition 10.** *Let  $U_1, U_2 \subseteq \overline{\mathbb{C}}$  be open sets. Then for every compact set  $N \subset U_1 \times U_2$  there exist compact sets  $N_1 \subseteq U_1$  and  $N_2 \subseteq U_2$  such that  $N \subseteq N_1 \times N_2$ .*

*Proof.* It is sufficient to take for  $N_1$  the image of the set  $N$  under the projection  $(\lambda, \mu) \mapsto \lambda$  onto the first coordinate, and for  $N_2$  the image of the set  $N$  under the projection  $(\lambda, \mu) \mapsto \mu$  onto the second coordinate.  $\square$

Let  $K$  be a closed subset of  $\overline{\mathbb{C}}^2$  or  $\overline{\mathbb{C}}$  and  $\mathbf{B}$  be a unital Banach algebra. We denote by  $\mathbf{O}(K, \mathbf{B})$  the set of all analytic<sup>1</sup> [54, 115] functions  $f: U \rightarrow \mathbf{B}$ , where  $U$  is an open neighbourhood of the set  $K$  (it is implied that the neighbourhood  $U$  may depend on  $f$ ). Two functions  $f_1: U_1 \rightarrow \mathbf{B}$  and  $f_2: U_2 \rightarrow \mathbf{B}$  are called *equivalent* if there exists an open neighbourhood  $U \subseteq U_1 \cap U_2$  of the set  $K$  such that  $f_1$  and  $f_2$  coincide on  $U$ , i. e.  $f_1(\lambda) = f_2(\lambda)$  for all  $\lambda \in U$ . It can be easily shown that this is really an equivalence relation. Thus, strictly speaking, elements of  $\mathbf{O}(K, \mathbf{B})$  are classes of equivalent functions. The notation  $\mathbf{O}(K, \mathbb{C})$  is abbreviated to  $\mathbf{O}(K)$ .

**Proposition 11.** *The set  $\mathbf{O}(K, \mathbf{B})$  is an algebra with respect to pointwise operations with the unit  $u(\lambda) = 1$ ,  $\lambda \in U \supset K$ .*

**Proposition 12.** (a) *For  $f \in \mathbf{O}(K, \mathbf{B})$ , the following conditions are equivalent: the function  $f$  is invertible in the algebra  $\mathbf{O}(K, \mathbf{B})$ ; the element  $f(\lambda) \in \mathbf{B}$  is invertible at all points  $\lambda \in K$ ; the element  $f(\lambda)$  is invertible at all points  $\lambda \in U$ , where  $U \supset K$  is some open set.* (b) *The spectrum of a function  $f \in \mathbf{O}(K, \mathbf{B})$  in the algebra  $\mathbf{O}(K, \mathbf{B})$  is given by the formula*

$$\bigcup_{\lambda \in K} \sigma(f(\lambda)).$$

We recall the definition of the natural topology on the algebra  $\mathbf{O}(K, \mathbf{B})$  [18, ch. 1, § 4.1].

For each open set  $U \supset K$ , we denote by  $\mathbf{O}(U, \mathbf{B})$  the linear space of all analytic functions  $f: U \rightarrow \mathbf{B}$ . We endow  $\mathbf{O}(U, \mathbf{B})$  with the *topology of compact convergence* [17, ch. X, § 3.6], [111, ch. III, § 3]. A fundamental system of neighbourhoods of zero in this topology is formed by the sets  $T(N, \delta) = \{f: \|f(N)\| < \delta\}$ , where  $N \subset U$  is compact and  $\delta > 0$ ; clearly, when  $N$  enlarges, the neighbourhood  $T(N, \delta)$  shrinks; therefore, it is enough to consider only those sets  $N$ , the interior of which contains  $K$ . There are evident canonical mappings  $g_U: \mathbf{O}(U, \mathbf{B}) \rightarrow \mathbf{O}(K, \mathbf{B})$ . The mappings  $g_U$  are not always injective. Nevertheless, by misuse of language, we will regard  $\mathbf{O}(U, \mathbf{B})$  as subspaces of the space  $\mathbf{O}(K, \mathbf{B})$ .

We endow  $\mathbf{O}(K, \mathbf{B})$  with the inductive topology [111, ch. II, § 6] induced by the mappings  $g_U$  (one may restrict himself to a decreasing sequence of open sets  $U \supset K$ ). A fundamental system of neighbourhoods of zero in  $\mathbf{O}(K, \mathbf{B})$  consists of all balanced, absorbent, and convex sets  $W \subseteq \mathbf{O}(K, \mathbf{B})$  such that the inverse image  $g_U^{-1}(W)$  is a

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<sup>1</sup>A function  $f: U \subset \overline{\mathbb{C}} \rightarrow \mathbf{B}$  is called *analytic at infinity* if  $f$  can be expanded in a power series  $f(\lambda) = \sum_{k=0}^{\infty} \frac{c_k}{\lambda^k}$  in a neighbourhood of infinity. A function  $f: U \subset \overline{\mathbb{C}}^2 \rightarrow \mathbb{C}$  is called *analytic at  $(\infty, \infty)$*  if  $f$  can be expanded in a power series  $f(\lambda, \mu) = \sum_{k,m=0}^{\infty} \frac{c_{km}}{\lambda^k \mu^m}$  in a neighbourhood of  $(\infty, \infty)$ . A function  $f: U \subset \overline{\mathbb{C}}^2 \rightarrow \mathbb{C}$  is called *analytic at  $(\lambda_*, \infty)$* ,  $\lambda_* \in \mathbb{C}$  if  $f$  can be expanded in a power series  $f(\lambda, \mu) = \sum_{k,m=0}^{\infty} \frac{c_{km}(\lambda - \lambda_*)^k}{\mu^m}$  in a neighbourhood of  $(\lambda_*, \infty)$ .

neighbourhood of zero in  $\mathbf{O}(U, \mathbf{B})$ . Thus, for all  $U \supset N \supset K$  such that the interior of the compact set  $N$  contains  $K$ , the inverse image  $g_U^{-1}(W)$  must contain the set  $T(N, \delta) = \{f: \|f(N)\| < \delta\} \subset \mathbf{O}(U, \mathbf{B})$ .

We recall [111, ch. 2, Theorem 6.1] that a linear mapping  $\varphi: \mathbf{O}(K, \mathbf{B}) \rightarrow \mathbb{E}$ , where  $\mathbb{E}$  is a Banach space, is continuous if and only if all the compositions  $\varphi \circ g_U: \mathbf{O}(U, \mathbf{B}) \rightarrow \mathbb{E}$ , where  $U \supset K$ , are continuous, i. e. for any neighbourhood  $W \subseteq \mathbb{E}$  of zero there exist a compact set  $N \subset U$  and a number  $\delta > 0$  such that the interior of  $N$  contains  $K$  and  $\varphi \circ g_U(T(N, \delta)) \subseteq W$ . We note that since  $\mathbb{E}$  is a Banach space, it is sufficient to restrict ourselves to the consideration of  $\varepsilon$ -neighbourhoods of zero for  $W$ . Below, by misuse of language, we denote  $\varphi \circ g_U$  by the abbreviated symbol  $\varphi$ .

**Proposition 13** ([69, Proposition 1.3]). *Let  $U_1 \subseteq \overline{\mathbb{C}}$  and  $U_2 \subseteq \overline{\mathbb{C}}$  be open sets. Then the natural image of the algebraic tensor product  $\mathbf{O}(U_1) \otimes \mathbf{O}(U_2)$  is everywhere dense in  $\mathbf{O}(U_1 \times U_2)$ .*

#### 4. PSEUDO-RESOLVENTS AND FUNCTIONAL CALCULUS

We call a mapping that converts functions to operators (or transformers) a *functional calculus*. Of course, the most interesting are functional calculi that possess special properties (for example, morphisms of algebras).

A *pseudo-resolvent* is a function that takes values in a Banach algebra and satisfies the Hilbert identity (2), like a resolvent. Every pseudo-resolvent generates a functional calculus (Theorems 25 and 26) which is a morphism of algebras and possesses the property of preserving the spectrum (Theorem 27).

Let  $\mathbf{B}$  be a Banach algebra and  $U \subseteq \mathbb{C}$  be a subset. A function (family)  $\lambda \mapsto R_\lambda$  defined on  $U$  and taking values in  $\mathbf{B}$  is called [67, ch. 5, § 2] a *pseudo-resolvent* if it satisfies the *Hilbert identity*

$$R_\lambda - R_\mu = -(\lambda - \mu)R_\lambda R_\mu, \quad \lambda, \mu \in U. \quad (3)$$

A pseudo-resolvent is called [9, p. 103] *maximal* if it cannot be extended to a larger set with the preservation of the Hilbert identity (3). Below (Theorem 16) we will see that every pseudo-resolvent can be extended to a unique maximal one. The domain  $\rho(R_{(\cdot)})$  of the maximal extension of a pseudo-resolvent  $R_{(\cdot)}$  is called a *regular set* of the original pseudo-resolvent. The complement  $\sigma(R_{(\cdot)})$  of the regular set  $\rho(R_{(\cdot)})$  is called [6], [9, p. 103] the *singular set*.

*Example 2.* The examples of pseudo-resolvents are: (a) the resolvent of an element of a unital Banach algebra (Proposition 14); (b) a constant function  $\lambda \mapsto N$ , where  $N \in \mathbf{B}$  is an arbitrary element whose square equals zero (Proposition 23); (c) in particular, the identically zero function; (d) the resolvent of a closed linear operator [67, Theorem 5.8.1]; (e) the resolvent of a linear relation [8, 9, 16, 20, 38, 57, 92]; this example is the most general, because every pseudo-resolvent is a resolvent of some linear relation [9, Theorem 5.2.4], [57, Proposition A.2.4]; (f) direct sums of pseudo-resolvents from the previous examples.

A simple example of a maximal pseudo-resolvent is given in the following proposition.

**Proposition 14** ([89, Proposition 17]). *The resolvent of an arbitrary element  $A \in \mathbf{B}$  is a maximal pseudo-resolvent, i. e. it cannot be extended to a set larger than  $\rho(A)$  with the preservation of the Hilbert identity (3).*



We note that identity (3) can be equivalently written in the form (here  $\mathbf{1}$  is an adjoint unit if the original algebra has no unit)

$$R_\lambda(\mathbf{1} + (\lambda - \mu)R_\mu) = R_\mu. \quad (4)$$

Below in this Section, we will adjoin a unit to  $\mathbf{B}$  when it has no unit.

**Proposition 15** ([67, Corollary 1 of Theorem 5.8.4]). *Let  $R_\lambda, R_\mu \in \mathbf{B}$  be two commuting elements that satisfy identity (3). Then the element  $\mathbf{1} + (\lambda - \mu)R_\mu \in \tilde{\mathbf{B}}$  is necessarily invertible.*

**Theorem 16** ([67, Theorem 5.8.6]). *Every pseudo-resolvent whose domain contains at least one point  $\mu \in \mathbb{C}$  can be extended to a maximal pseudo-resolvent; this extension is unique. The domain of the maximal extension is the set of all  $\lambda \in \mathbb{C}$  such that the element  $\mathbf{1} + (\lambda - \mu)R_\mu$  is invertible in  $\tilde{\mathbf{B}}$ . This extension can be defined by the formula*

$$R_\lambda = R_\mu(\mathbf{1} + (\lambda - \mu)R_\mu)^{-1} = (\mathbf{1} + (\lambda - \mu)R_\mu)^{-1}R_\mu. \quad (5)$$

We will denote the original pseudo-resolvent and its continuation to a maximal pseudo-resolvent by the same symbol  $R_{(\cdot)}$ . Moreover, we will generally assume that all pseudo-resolvents under consideration are already extended to maximal pseudo-resolvents.

**Corollary 17** ([67, Theorem 5.8.2], [20, ch. 6, § 1]). *The domain of a maximal pseudo-resolvent is an open set and the maximal pseudo-resolvent is an analytic function (with values in  $\mathbf{B}$ ).<sup>2</sup> More precisely, in a neighbourhood of any point  $\mu \in \rho(R_{(\cdot)})$ , the maximal pseudo-resolvent admits the power series expansion*

$$R_\lambda = \sum_{n=0}^{\infty} (\mu - \lambda)^n R_\mu^{n+1}.$$

**Proposition 18** ([67, Theorem 5.8.3]). *A pseudo-resolvent  $R_{(\cdot)}$  in a Banach algebra  $\mathbf{B}$  is a resolvent of some element  $A \in \mathbf{B}$  if and only if  $\mathbf{B}$  is unital and the element  $R_\mu$  is invertible for at least one (and, consequently, for all)  $\mu \in \rho(R_{(\cdot)})$ . A pseudo-resolvent  $R_{(\cdot)}$  in  $\mathbf{B}(X)$ , where  $X$  is a Banach space, is a resolvent of some unbounded operator  $A : D(A) \subset X \rightarrow X$  if and only if the operator  $R_\mu : X \rightarrow \text{Im } R_\mu$  is invertible for at least one (and, consequently, for all)  $\mu \in \rho(R_{(\cdot)})$ . In this case  $A = \lambda \mathbf{1} - (R_\lambda)^{-1}$  for all  $\lambda \in \rho(R_{(\cdot)})$ .*

In a similar way, a linear relation can also be recovered from the value of its resolvent at one point. Thus, the resolvent contains all information about a linear relation or an operator that generates it. On the other hand, the conditions on unbounded operators and linear relations are often imposed in terms of their resolvents (the nonemptiness of the resolvent set, the estimate of decay rate at infinity etc.). Besides, functions of linear relations and unbounded operators are often expressed directly via their resolvents. For this reason, the resolvent can be considered as a more fundamental object than an operator or relation that generates it. This is the viewpoint we adhere to in this article.

We fix a pseudo-resolvent  $R_{(\cdot)}$ . We denote by  $\mathbf{B}_R$  the smallest closed subalgebra of the algebra  $\mathbf{B}$  that contains all elements  $R_\lambda$ ,  $\lambda \in \rho(R_{(\cdot)})$ , of the extension of the pseudo-resolvent  $R_{(\cdot)}$  to a maximal pseudo-resolvent.

**Proposition 19** ([89, Proposition 21]). *The algebra  $\mathbf{B}_R$  coincides with the closure of the linear span of the family of all elements  $R_\lambda$ ,  $\lambda \in \rho(R_{(\cdot)})$ , and is commutative.*

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<sup>2</sup>We accept that the domain of an analytic function may be disconnected.

If the algebra  $\mathbf{B}_R$  does not contain the unit of the algebra  $\mathbf{B}$  (this is certainly the case if  $\mathbf{B}$  is not unital), then we will, as usual, denote by  $\widetilde{\mathbf{B}}_R$  the algebra  $\mathbf{B}_R$  with an adjoint unit from  $\mathbf{B}^3$  (or the algebra  $\widetilde{\mathbf{B}}$  with an adjoint unit). If  $\mathbf{B}_R$  contains the unit of the algebra  $\mathbf{B}$ , the symbol  $\widetilde{\mathbf{B}}_R$  is understood to be  $\mathbf{B}_R$ .

**Proposition 20** ([89, Theorem 22]). *The subalgebra  $\widetilde{\mathbf{B}}_R$  is commutative and full.*

**Proposition 21.** *If a character  $\chi$  of the algebra  $\mathbf{B}_R$  equals zero at least at one element  $R_\mu$ ,  $\mu \in \rho(R_{(\cdot)})$ , then it is identically equal to zero on  $\mathbf{B}_R$ , i. e. coincides with  $\chi_0$ .*

*Proof.* The proof follows from formula (5) and the description of  $\mathbf{B}_R$  (Proposition 19) as the closure of the linear span of the family  $R_\lambda$ ,  $\lambda \in \rho(R_{(\cdot)})$ .  $\square$

We say that a sequence of maximal pseudo-resolvents  $R_{n,(\cdot)}$  converges to a maximal pseudo-resolvent  $R_{(\cdot)}$  if there exists a point  $\mu \in \mathbb{C}$  such that all the pseudo-resolvents  $R_{n,(\cdot)}$  are defined at  $\mu$  (for  $n$  sufficiently large) and the sequence  $R_{n,\mu}$  converges to  $R_\mu$  in norm, cf. [75, Theorem 2.25]. The following lemma shows that this definition does not depend on the choice of the point  $\mu \in \mathbb{C}$ .

**Lemma 22.** *Let a sequence  $R_{n,(\cdot)}$  of maximal pseudo-resolvents converge to a maximal pseudo-resolvent  $R_{(\cdot)}$  at a point  $\mu \in \mathbb{C}$  (it is assumed that the pseudo-resolvents  $R_{n,\mu}$  are defined at the point  $\mu$  for all  $n$  large enough). Then for any point  $\lambda \in \rho(R_{(\cdot)})$ , the sequence  $R_{n,\lambda}$  is defined for all  $n$  large enough and converges to  $R_\lambda$  in norm.*

*Moreover, given a compact set  $\Gamma \subset \rho(R_{(\cdot)})$ , the elements  $R_{n,\lambda}$  are defined for  $n$  large enough at all  $\lambda \in \Gamma$  and converges to  $R_\lambda$  uniformly with respect to  $\lambda \in \Gamma$ .*

*Proof.* Let  $R_{n,\mu}$  converge to  $R_\mu$ . By Theorem 16, the element  $\mathbf{1} + (\lambda - \mu)R_\mu$  is invertible for all  $\lambda \in \Gamma$ . Since the function  $\lambda \mapsto \mathbf{1} + (\lambda - \mu)R_\mu$  is continuous, from Theorem 1 it follows that

$$\min_{\lambda \in \Gamma} \|\mathbf{1} + (\lambda - \mu)R_\mu\|^{-1} > 0.$$

Since  $R_{n,\mu}$  converges to  $R_\mu$ , the sequence  $\lambda \mapsto \mathbf{1} + (\lambda - \mu)R_{n,\mu}$  converges to  $\lambda \mapsto \mathbf{1} + (\lambda - \mu)R_\mu$  uniformly with respect to  $\lambda \in \Gamma$ . Therefore, again by Theorem 1, the elements  $\mathbf{1} + (\lambda - \mu)R_{n,\mu}$  are invertible for all  $\lambda \in \Gamma$  provided  $n$  is large enough; in this case, by estimate (1), the inverses also converge uniformly. It remains to apply formula (5).  $\square$

We note that the limit of a sequence of resolvents of bounded operators can be the resolvent of a non-bounded operator, see [105, Lemma 7].

**Proposition 23** ([67, Theorem 5.9.2]). *Let a pseudo-resolvent  $R_{(\cdot)}$  admit an analytic continuation in a neighbourhood of the point  $\infty$ .<sup>4</sup> Then there exist elements  $P, A, N \in \mathbf{B}_R$  such that*

$$N^2 = \mathbf{0}, \quad P^2 = P, \quad AP = PA = A, \quad NP = PN = \mathbf{0}$$

*and the expansion of the pseudo-resolvent into the Laurent series with centre  $\infty$  has the form*

$$R_\lambda = -N + \frac{P}{\lambda} + \frac{A}{\lambda^2} + \frac{A^2}{\lambda^3} + \frac{A^3}{\lambda^4} + \dots \quad (6)$$

<sup>3</sup>Adjoining  $\mathbf{1} \in \mathbf{B}$  to  $\mathbf{B}_R$  we obtain a closed subalgebra because the sum of a closed subspace and a one-dimensional subspace is a closed subspace.

<sup>4</sup>We recall [54, p. 107] that the possibility of an analytic continuation of  $f$  in a neighbourhood of the point  $\infty$  is equivalent to the existence of a bounded analytic continuation of  $f$  in a deleted neighbourhood of  $\infty$ .

We call the *extended regular set*  $\bar{\rho}(R_{(\cdot)}) \subseteq \overline{\mathbb{C}}$  of a pseudo-resolvent  $R_{(\cdot)}$  (in the algebra  $\mathbf{B}$ ) either the regular set  $\rho(R_{(\cdot)})$  or the union  $\rho(R_{(\cdot)}) \cup \{\infty\}$ ; more precisely, we add the point  $\infty$  to  $\bar{\rho}(R_{(\cdot)})$  if the algebra  $\mathbf{B}$  is unital, the regular set  $\rho(R_{(\cdot)})$  contains a (deleted) neighbourhood of  $\infty$ , and  $\lim_{\lambda \rightarrow \infty} \lambda R_\lambda = \mathbf{1}$ . We call the *extended singular set* of the pseudo-resolvent the complement  $\bar{\sigma}(R_{(\cdot)}) = \overline{\mathbb{C}} \setminus \bar{\rho}(R_{(\cdot)})$  of the extended regular set.

**Proposition 24.** *The following properties of a maximal pseudo-resolvent are equivalent:*

- (a)  $\infty \in \bar{\rho}(R_{(\cdot)})$ ;
- (b) *the maximal pseudo-resolvent is the resolvent of some element  $A \in \mathbf{B}_R$  (see Proposition 18);*
- (c) *the algebra  $\mathbf{B}$  is unital and the subalgebra  $\mathbf{B}_R$  contains the unit of the algebra  $\mathbf{B}$ .*

*Proof.* The equivalence of (a) and (b) is proved in [89, Proposition 23].

Let assumption (b) be fulfilled, i. e.  $R_\lambda = (\lambda \mathbf{1} - A)^{-1}$ , where  $A \in \mathbf{B}_R$ . Then, by virtue of Theorem 1, the power series expansion

$$(\lambda \mathbf{1} - A)^{-1} = \frac{\mathbf{1}}{\lambda} + \frac{A}{\lambda^2} + \frac{A^2}{\lambda^3} + \dots$$

holds in a neighbourhood of infinity, which shows that  $\mathbf{1} \in \mathbf{B}_R$ , i. e., assumption (c) holds.

Let assumption (c) be fulfilled, i. e. the subalgebra  $\mathbf{B}_R$  contain the unit of the algebra  $\mathbf{B}$ . There are no identically zero characters on a unital commutative algebra, because  $\chi(\mathbf{1}) = 1$ . Therefore, by Proposition 21,  $\chi(R_\mu) \neq 0$  for all  $\chi \in \tilde{X}(\mathbf{B}_R)$  and  $\mu \in \rho(R_{(\cdot)})$ . Hence, by Theorem 6, all values  $R_\mu$  of the pseudo-resolvent are invertible. Then, by virtue of Proposition 18, the pseudo-resolvent is the resolvent of some element  $A \in \mathbf{B}_R$ , i. e. assumption (b) holds.  $\square$

Below in this Section, we assume that  $X$  is a Banach space and we are given a maximal pseudo-resolvent  $R_{(\cdot)}$  in  $\mathbf{B}(X)$ .

Let  $\sigma$  and  $\Sigma$  be two disjoint closed subsets of  $\overline{\mathbb{C}}$ . A contour  $\Gamma$  is called [67, ch. V, § 5.2] an *oriented envelope* of the set  $\sigma$  *with respect* to the set  $\Sigma$  if  $\Gamma$  is an oriented boundary of an open set  $U$  that contains  $\sigma$  and is disjoint from  $\Sigma$ . Thus,  $\Gamma$  surrounds the set  $\sigma$  in the counterclockwise direction and surrounds the set  $\Sigma$  in the clockwise direction.

**Theorem 25.** *Assume that  $\infty \notin \bar{\sigma}(R_{(\cdot)})$ . We define the mapping  $\varphi: \mathbf{O}(\sigma(R_{(\cdot)})) \rightarrow \mathbf{B}_R$  by the formula*

$$\varphi(f) = \frac{1}{2\pi i} \int_{\Gamma} f(\lambda) R_\lambda d\lambda, \quad (7)$$

where  $\Gamma$  (see left fig. 1) is an oriented envelope of the singular set  $\sigma(R_{(\cdot)})$  with respect to the point  $\infty$  and the complement of the domain of the function  $f$ . We assert that  $\varphi$  is a continuous morphism of unital algebras.

The morphism  $\varphi$  maps the function  $u(\lambda) = 1$  to the identity operator  $\mathbf{1}_X$ . The function  $v_1(\lambda) = \lambda$  is mapped by  $\varphi$  to the operator  $A \in \mathbf{B}(X)$  that generates the maximal pseudo-resolvent  $R_{(\cdot)}$  in accordance with Proposition 24; and the function  $r_{\lambda_0}(\lambda) = \frac{1}{\lambda_0 - \lambda}$ , where  $\lambda_0 \in \rho(R_{(\cdot)})$ , is mapped by  $\varphi$  to  $R_{\lambda_0}$ .

*Proof.* The proof is analogous to that of the theorem on analytic functional calculus for bounded operators [18, ch. 1, § 4, Theorem 3], [67, Theorem 5.2.5], [110, Theorem 10.27].  $\square$

When it is desirable to stress that the functional calculus  $\varphi$  considered in Theorem 25 is generated by the resolvent of an operator  $A \in \mathbf{B}(X)$ , we will use the notation  $R_{A,\lambda}$  instead of  $R_\lambda$ , the notation  $\varphi_A$  instead of  $\varphi$ , and the notation  $f(A)$  instead of  $\varphi(f)$ .

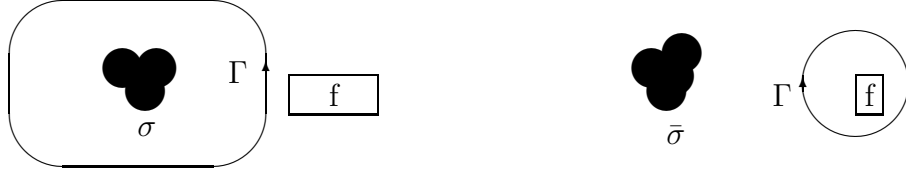


FIGURE 1. The contour  $\Gamma$  is an oriented envelope of the set  $\sigma$  with respect to  $\infty$  and the set  $\boxed{f}$  (left); the contour  $\Gamma$  is an oriented envelope of the set  $\bar{\sigma}$  and the point  $\infty$  with respect to the set  $\boxed{f}$  (right).

**Theorem 26.** Assume that  $\infty \in \bar{\sigma}(R_{(\cdot)})$ . We define the mapping  $\varphi: \mathbf{O}(\bar{\sigma}(R_{(\cdot)})) \rightarrow \tilde{\mathbf{B}}_R$  by the formula

$$\varphi(f) = \frac{1}{2\pi i} \int_{\Gamma} f(\lambda) R_{\lambda} d\lambda + f(\infty) \mathbf{1}, \quad (8)$$

where  $\Gamma$  is an oriented envelope of the extended singular set  $\bar{\sigma}(R_{(\cdot)})$  with respect to the complement of the domain of  $f$  (see the right fig. 1). We assert that  $\varphi$  is a continuous morphism of unital algebras.

The morphism  $\varphi$  maps the function  $u(\lambda) = 1$  to the identity operator  $\mathbf{1}$ . The function  $r_{\lambda_0}(\lambda) = \frac{1}{\lambda_0 - \lambda}$ , where  $\lambda_0 \in \rho(R_{(\cdot)})$ , is mapped by  $\varphi$  to  $R_{\lambda_0}$ .

*Proof.* The proof is analogous to that of the theorem on analytic functional calculus for unbounded operators [67, Theorem 5.11.2].  $\square$

When it is desirable to stress that the functional calculus  $\varphi$  considered in Theorem 26 is generated by a pseudo-resolvent  $R_{(\cdot)}$ , we will use the notation  $\varphi_{R_{(\cdot)}}$  instead of  $\varphi$  and the notation  $f(R_{(\cdot)})$  instead of  $\varphi(f)$ .

A unified notation for formulae (7) and (8) is suggested in [67]:

$$\varphi(f) = \frac{1}{2\pi i} \int_{\Gamma} f(\lambda) R_{\lambda} d\lambda + \delta f(\infty) \mathbf{1},$$

where  $\delta = 0$  if  $\Gamma$  does not enclose  $\infty$ , and  $\delta = 1$  if  $\Gamma$  encloses  $\infty$ .

The following theorem is a version of the spectral mapping theorem for the case of pseudo-resolvents.

**Theorem 27.** For any function  $f \in \mathbf{O}(\bar{\sigma}(R_{(\cdot)}))$  we have the equality

$$\sigma_{\tilde{\mathbf{B}}}(\varphi(f)) = \sigma_{\tilde{\mathbf{B}}_R}(\varphi(f)) = \{f(\lambda): \lambda \in \bar{\sigma}(R_{(\cdot)})\}.$$

*Proof.* The proof is analogous to that of the spectral mapping theorem for unbounded operators [67, 5.12.1] and for linear relations [9, 5.2.17].  $\square$

An analytic functional calculus for bounded operators was created in [34, 35, 123]. It was carried over to unbounded operators in [33, 67] and to linear relations in [8, 9, 20, 38, 57].

## 5. EXTENDED TENSOR PRODUCTS

The notion of an extended tensor product is a generalization of the notion of a completion of an algebraic tensor product with respect to a uniform cross-norm [28, 31, 55, 64, 112]. It enables one to extend some constructions which are natural for the usual tensor products to supplementary applications. We recall an example which is the most

important for our applications. It is known (see, e. g., [53]) that in the case of finite-dimensional Banach spaces  $X$  and  $Y$ , the space  $\mathbf{B}(Y, X)$  can be identified with the tensor product  $X \otimes Y^*$ . If  $X$  and  $Y$  are infinite-dimensional, then  $X \otimes Y^*$  corresponds only to the subspace of  $\mathbf{B}(Y, X)$  consisting of operators that have a finite-dimensional image. Therefore the completion of  $X \otimes Y^*$  with respect to any reasonable norm cannot coincide with the whole  $\mathbf{B}(Y, X)$ . Nevertheless,  $\mathbf{B}(Y, X)$  can be represented (see example 3(e) below) as an extended tensor product  $X \boxtimes Y^*$  which enables one to treat it almost as a usual tensor product. The exposition in this Section is based on [89].

We denote by  $X \otimes Y$  the usual tensor product of linear spaces  $X$  and  $Y$ . In order to distinguish  $X \otimes Y$  from its extensions, we call  $X \otimes Y$  an *algebraic tensor product*.

Let  $X$  and  $Y$  be Banach spaces. A norm  $\alpha(\cdot) = \|\cdot\|_\alpha$  on  $X \otimes Y$  is called a *cross-norm* if

$$\|x \otimes y\|_\alpha = \|x\| \cdot \|y\|$$

for all  $x \in X$  and  $y \in Y$ . We denote by  $X \overline{\otimes}_\alpha Y$  the completion of the tensor product  $X \otimes Y$  by the cross-norm  $\alpha$ .

Every element  $v^* = \sum_{l=1}^m x_l^* \otimes y_l^* \in X^* \otimes Y^*$  induces the linear functional

$$v^*: \sum_{k=1}^n x_k \otimes y_k \mapsto \sum_{l=1}^m \sum_{k=1}^n \langle x_k, x_l^* \rangle \cdot \langle y_k, y_l^* \rangle \quad (9)$$

on the space  $X \otimes Y$ . We define the norm  $\alpha^*$  on  $X^* \otimes Y^*$ , *conjugate to the cross-norm  $\alpha$* , by the formula

$$\|v^*\|_{\alpha^*} = \sup\{ |\langle v, v^* \rangle| : v \in X \otimes Y, \|v\|_\alpha \leq 1 \}.$$

A cross-norm  $\alpha$  is called *\*-uniform* if  $\alpha^*$  is finite and is a cross-norm.

The space  $\mathbf{B}(X) \otimes \mathbf{B}(Y)$  has the natural structure of an algebra. Every element  $T = \sum_{l=1}^m A_l \otimes B_l \in \mathbf{B}(X) \otimes \mathbf{B}(Y)$  induces the linear operator

$$T: \sum_{k=1}^n x_k \otimes y_k \mapsto \sum_{l=1}^m \sum_{k=1}^n (A_l x_k) \otimes (B_l y_k)$$

in  $X \otimes Y$ . A cross-norm  $\alpha$  on the space  $X \otimes Y$  induces the norm  $\tilde{\alpha}$  of the operator  $T \in \mathbf{B}(X) \otimes \mathbf{B}(Y)$  by the formula

$$\|T\|_{\tilde{\alpha}} = \sup\{ \|Tv\| : v \in X \otimes Y, \|v\|_\alpha \leq 1 \}.$$

A cross-norm  $\alpha$  is called [112] *uniform* if  $\tilde{\alpha}$  is finite and is a cross-norm. Every uniform cross-norm is \*-uniform, see [118].

Let  $X$  and  $Y$  be Banach spaces. We call an *extended tensor product* [89] of  $X$  and  $Y$  a collection consisting of three objects: a Banach space  $X \boxtimes Y$  (which we briefly refer to as the extended tensor product) and two (not necessarily closed) full unital subalgebras  $\mathbf{B}_0(X)$  and  $\mathbf{B}_0(Y)$  of the algebras  $\mathbf{B}(X)$  and  $\mathbf{B}(Y)$  respectively that satisfy assumptions (A), (B), and (C) listed below.

- (A) We are given a linear mapping  $j$  from the algebraic tensor product  $X \otimes Y$  to  $X \boxtimes Y$ . In the sequel, we denote  $j(x \otimes y)$  by the symbol  $x \boxtimes y$ . It is assumed that

$$\|x \boxtimes y\|_{X \boxtimes Y} = \|x\|_X \cdot \|y\|_Y \quad (10)$$

for all  $x \in X$  and  $y \in Y$ .

- (B) We are given a linear mapping  $J$  from the algebraic tensor product  $X^* \otimes Y^*$  to  $(X \boxtimes Y)^*$ . In the sequel, we denote  $J(x^* \otimes y^*)$  by the symbol  $x^* \boxtimes y^*$ . It is assumed that

$$\langle x \boxtimes y, x^* \boxtimes y^* \rangle = \langle x, x^* \rangle \langle y, y^* \rangle \quad (11)$$

for all  $x^* \in X^*$ ,  $y^* \in Y^*$ ,  $x \in X$ , and  $y \in Y$ , and

$$\|x^* \boxtimes y^*\|_{(X \boxtimes Y)^*} = \|x^*\|_{X^*} \cdot \|y^*\|_{Y^*} \quad (12)$$

for all  $x^* \in X^*$  and  $y^* \in Y^*$ .

(C) We are given a morphism  $\mathfrak{J}$  of unital algebras from the algebraic tensor product  $\mathbf{B}_0(X) \otimes \mathbf{B}_0(Y)$  to  $\mathbf{B}(X \boxtimes Y)$ . In the sequel, we denote  $\mathfrak{J}(A \otimes B)$  by the symbol  $A \boxtimes B$ . It is assumed that

$$(A \boxtimes B)(x \boxtimes y) = (Ax) \boxtimes (By) \quad (13)$$

for all  $A \in \mathbf{B}_0(X)$ ,  $B \in \mathbf{B}_0(Y)$ ,  $x \in X$ , and  $y \in Y$ , and

$$(A \boxtimes B)^*(x^* \boxtimes y^*) = (A^*x^*) \boxtimes (B^*y^*) \quad (14)$$

for all  $A \in \mathbf{B}_0(X)$ ,  $B \in \mathbf{B}_0(Y)$ ,  $x^* \in X^*$ , and  $y^* \in Y^*$ , and

$$\|A \boxtimes B\|_{\mathbf{B}(X \boxtimes Y)} = \|A\|_{\mathbf{B}(X)} \cdot \|B\|_{\mathbf{B}(Y)} \quad (15)$$

for all  $A \in \mathbf{B}_0(X)$  and  $B \in \mathbf{B}_0(Y)$ .

*Example 3.* We recall [89] some examples of extended tensor products.

(a) Let  $\alpha$  be a cross-norm on an algebraic tensor product  $X \otimes Y$ . We take for  $X \boxtimes Y$  the completion  $X \overline{\otimes}_\alpha Y$  of the space  $X \otimes Y$  with respect to the cross-norm  $\alpha$ , and we take for  $\mathbf{B}_0(X)$  and  $\mathbf{B}_0(Y)$  the algebras  $\mathbf{B}(X)$  and  $\mathbf{B}(Y)$  respectively. In such a case, assumption (12) means that the cross-norm  $\alpha$  is *\*-uniform*, and assumption (15) means that the cross-norm  $\alpha$  is *uniform*.

(b) Let  $X$  and  $Y$  be Banach spaces. We denote by  $\mathbf{K}(X, Y)$  the Banach space of all bilinear forms  $K: X \times Y \rightarrow \mathbb{C}$  that are bounded with respect to the norm  $\|K\| = \sup\{|K(x, y)|: \|x\| \leq 1, \|y\| \leq 1\}$ . In order to represent  $\mathbf{K}(X, Y)$  as an extended tensor product  $X^* \boxtimes Y^*$ , we take for  $\mathbf{B}_0(X^*)$  and  $\mathbf{B}_0(Y^*)$  the subalgebras of algebras  $\mathbf{B}(X^*)$  and  $\mathbf{B}(Y^*)$  consisting of all operators that have a preconjugate. We define the mappings  $j$ ,  $J$ , and  $\mathfrak{J}$  by the rules (extended by linearity)

$$\begin{aligned} [x^* \boxtimes y^*](x, y) &= \langle x, x^* \rangle \langle y, y^* \rangle, \\ \langle x^{**} \boxtimes y^{**}, K \rangle &= \overline{K}(x^{**}, y^{**}), \\ [(A \boxtimes B)K](x, y) &= K(A^0x, B^0y), \end{aligned}$$

where  $\overline{K}$  is the canonical extension [7] of  $K$  to  $X^{**} \times Y^{**}$ .

(c) Let  $X$  and  $Y$  be Banach spaces, and  $X \overline{\otimes}_\alpha Y$  be a completion of the space  $X \otimes Y$  with respect to a uniform cross-norm  $\alpha$ . The conjugate space  $(X \overline{\otimes}_\alpha Y)^*$  can be regarded as an extended tensor product  $X^* \boxtimes Y^*$  if one takes for  $\mathbf{B}_0(X^*)$  and  $\mathbf{B}_0(Y^*)$  the subalgebras of the algebras  $\mathbf{B}(X^*)$  and  $\mathbf{B}(Y^*)$  consisting of all operators that have a preconjugate. We notice that this example is a generalization of the previous one, since  $\mathbf{K}(X, Y) \cong (X \overline{\otimes}_\pi Y)^*$ , where  $\pi$  is the largest cross-norm [28, 64, 112].

We define  $j: X^* \otimes Y^* \rightarrow X^* \boxtimes Y^* = (X \overline{\otimes}_\alpha Y)^*$  as the canonical embedding (9).

Next, we define  $J: X^{**} \otimes Y^{**} \rightarrow (X^* \boxtimes Y^*)^* = (X \overline{\otimes}_\alpha Y)^{**}$ . To this end, we assign to each functional  $w^* \in X^* \boxtimes Y^* = (X \overline{\otimes}_\alpha Y)^*$  the bilinear form  $K_{w^*}(x, y) = \langle x \otimes y, w^* \rangle$  on  $X \times Y$ . For  $\sum_{k=1}^n x_k^{**} \otimes y_k^{**} \in X^{**} \otimes Y^{**}$ , we set

$$\left\langle J\left(\sum_{k=1}^n x_k^{**} \otimes y_k^{**}\right), w^* \right\rangle = \sum_{k=1}^n \overline{K_{w^*}}(x_k^{**}, y_k^{**}),$$

where  $\overline{K_{w^*}}$  is the canonical extension of the bilinear form  $K_{w^*}$  to  $X^{**} \times Y^{**}$ .

We define the operator  $\mathfrak{J}(\sum_{k=1}^n A_k \otimes B_k) \in \mathbf{B}((X \overline{\otimes}_\alpha Y)^*)$  as the conjugate of the operator  $\sum_{k=1}^n A_k^0 \otimes B_k^0: X \overline{\otimes}_\alpha Y \rightarrow X \overline{\otimes}_\alpha Y$ .

(d) Let  $X = L_\infty[a, b]$  and  $Y = L_\infty[c, d]$ . By example (c), the space  $L_\infty[a, b] \times [c, d]$  can be regarded as the extended tensor product  $L_\infty[a, b] \boxtimes L_\infty[c, d]$  (we recall that the space  $L_\infty[a, b]$  is conjugate of the space  $L_1[a, b]$ ). We notice that one should take for  $\mathbf{B}_0(X)$  and  $\mathbf{B}_0(Y)$  the subalgebras of the algebras  $\mathbf{B}(X)$  and  $\mathbf{B}(Y)$  consisting of all operators that have a preconjugate.

(e) Let  $X$  and  $Y$  be Banach spaces. We represent the space  $\mathbf{B}(Y, X)$  as an extended tensor product  $X \boxtimes Y^*$ . To this end, we take for  $\mathbf{B}_0(X)$  the whole algebra  $\mathbf{B}(X)$  and we take for  $\mathbf{B}_0(Y^*)$  the subalgebra of the algebra  $\mathbf{B}(Y^*)$  consisting of all operators that have a preconjugate. We define the mappings  $j$ ,  $J$ , and  $\mathfrak{J}$  by the rules (extended by linearity)

$$\begin{aligned} (x \boxtimes y^*)y &= x \langle y, y^* \rangle, \\ \langle U, x^* \boxtimes y^{**} \rangle &= \langle y^{**}, U^* x^* \rangle, \\ (A \boxtimes B)U &= AUB^0. \end{aligned}$$

Note that in this example the subalgebra  $\mathbf{B}_0(Y^*)$  can be thought of as  $\mathbf{B}(Y)$ , but the action of  $\mathbf{B}(Y)$  on  $U \in \mathbf{B}(Y, X)$  should be understood as contravariant, i. e.,

$$(A_1 \boxtimes B_1)((A_2 \boxtimes B_2)U) = A_1 A_2 U B_2 B_1.$$

Below in this Section, we assume that we are given an extended tensor product  $X \boxtimes Y$  of Banach spaces  $X$  and  $Y$ , and a pseudo-resolvent  $R_{(\cdot)}$  in the algebra  $\mathbf{B}_0(Y)$ .

**Theorem 28** ([89, Theorem 26]). *Assume that  $\infty \notin \bar{\sigma}(R_{(\cdot)})$ . We define the mapping  $\Phi: \mathbf{O}(\sigma(R_{(\cdot)}), \mathbf{B}_0(X)) \rightarrow \mathbf{B}(X \boxtimes Y)$  by the formula*

$$\Phi(F) = \frac{1}{2\pi i} \int_{\Gamma} F(\lambda) \boxtimes R_\lambda d\lambda, \quad (16)$$

where  $\Gamma$  is an oriented envelope of the singular set  $\sigma(R_{(\cdot)})$  of the pseudo-resolvent with respect to the point  $\infty$  and the complement of the domain of  $F$ . We assert that  $\Phi$  is a continuous morphism of unital algebras.

For all  $A \in \mathbf{B}_0(X)$  and  $h \in \mathbf{O}(\sigma(R_{(\cdot)}))$  the morphism  $\Phi$  maps the function  $F(\lambda) = Ah(\lambda)$  to the operator  $A \boxtimes \varphi(h)$ , where  $\varphi$  is defined as in Theorem 25.

We stress that the function  $F$  in (16) takes its values in  $\mathbf{B}_0(X)$ , but not in  $\mathbb{C}$ .

**Theorem 29** ([89, Theorem 27]). *Assume that  $\infty \in \bar{\sigma}(R_{(\cdot)})$ . We define the mapping  $\Phi: \mathbf{O}(\bar{\sigma}(R_{(\cdot)}), \mathbf{B}_0(X)) \rightarrow \mathbf{B}(X \boxtimes Y)$  by the formula*

$$\Phi(F) = \frac{1}{2\pi i} \int_{\Gamma} F(\lambda) \boxtimes R_\lambda d\lambda + F(\infty) \boxtimes \mathbf{1}, \quad (17)$$

where  $\Gamma$  is an oriented envelope of the extended singular set  $\bar{\sigma}(R_{(\cdot)})$  of the pseudo-resolvent with respect to the complement of the domain of  $F$ . We assert that  $\Phi$  is a continuous morphism of unital algebras.

For all  $A \in \mathbf{B}_0(X)$  and  $h \in \mathbf{O}(\bar{\sigma}(R_{(\cdot)}))$  the morphism  $\Phi$  maps the function  $F(\lambda) = Ah(\lambda)$  to the operator  $A \boxtimes \varphi(h)$ , where  $\varphi$  is defined as in Theorem 26.

**Theorem 30** ([89, Theorem 41]). *Let  $F \in \mathbf{O}(\bar{\sigma}(R_{(\cdot)}), \mathbf{B}_0(X))$ . We define the operator  $\Phi(F)$  by formula (16) if  $\infty \notin \bar{\sigma}(R_{(\cdot)})$ ; and we define the operator  $\Phi(F)$  by formula (17) if  $\infty \in \bar{\sigma}(R_{(\cdot)})$ . We assert that the operator  $\Phi(F): X \boxtimes Y \rightarrow X \boxtimes Y$  is not invertible if and only if for some  $\lambda \in \bar{\sigma}(R_{(\cdot)})$  the operator  $F(\lambda) \in \mathbf{B}_0(X)$  is not invertible.*

**Theorem 31** ([89, Theorem 42]). *Let  $F \in \mathbf{O}(\bar{\sigma}(R_{(\cdot)}), \mathbf{B}_0(X))$ . We define the operator  $\Phi(F)$  by formula (16) if  $\infty \notin \bar{\sigma}(R_{(\cdot)})$ ; and we define the operator  $\Phi(F)$  by formula (17) if  $\infty \in \bar{\sigma}(R_{(\cdot)})$ . We assert that the spectrum of the operator  $\Phi(F): X \boxtimes Y \rightarrow X \boxtimes Y$  is given by the formula*

$$\sigma[\Phi(F)] = \bigcup_{\lambda \in \bar{\sigma}(R_{(\cdot)})} \sigma(F(\lambda)).$$

## 6. FUNCTIONAL CALCULUS $\varphi_1 \boxtimes \varphi_2$

In this Section, we discuss the product  $\varphi_1 \boxtimes \varphi_2$  of functional calculi  $\varphi_1$  and  $\varphi_2$  that were defined in Section 4; it acts in the extended tensor product  $X \boxtimes Y$ . Keeping in mind the space  $\mathbf{B}(Y, X)$  (see Example 3(e)) as the main example of an extended tensor product, we call *transformators* operators acting in  $X \boxtimes Y$ .

In this Section, we assume that we are given an extended tensor product  $X \boxtimes Y$  of Banach spaces  $X$  and  $Y$ , and we are given pseudo-resolvents  $R_{1,(\cdot)}$  and  $R_{2,(\cdot)}$  in the algebras  $\mathbf{B}_0(X)$  and  $\mathbf{B}_0(Y)$  respectively.

**Theorem 32.** *Assume that  $\infty \notin \bar{\sigma}(R_{1,(\cdot)})$  and  $\infty \notin \bar{\sigma}(R_{2,(\cdot)})$ . We define the mapping  $\varphi_1 \boxtimes \varphi_2: \mathbf{O}(\sigma(R_{1,(\cdot)}) \times \sigma(R_{2,(\cdot)})) \rightarrow \mathbf{B}(X \boxtimes Y)$  by the formula*

$$(\varphi_1 \boxtimes \varphi_2)f = \frac{1}{(2\pi i)^2} \int_{\Gamma_1} \int_{\Gamma_2} f(\lambda, \mu) R_{1,\lambda} \boxtimes R_{2,\mu} d\mu d\lambda,$$

where  $\Gamma_1$  and  $\Gamma_2$  are oriented envelopes of the singular sets  $\sigma(R_{1,(\cdot)})$  and  $\sigma(R_{2,(\cdot)})$  with respect to the point  $\infty$  and the complements  $\bar{\mathbb{C}} \setminus U_1$  and  $\bar{\mathbb{C}} \setminus U_2$ ; here  $U_1 \times U_2$  is an open neighbourhood of the set  $\sigma(R_{1,(\cdot)}) \times \sigma(R_{2,(\cdot)})$  that lies in the domain of the function  $f$  (see Proposition 9). We assert that  $\varphi_1 \boxtimes \varphi_2$  is a continuous morphism of unital algebras.

For all  $g \in \mathbf{O}(\sigma(R_{1,(\cdot)}))$  and  $h \in \mathbf{O}(\sigma(R_{2,(\cdot)}))$  the morphism  $\varphi_1 \boxtimes \varphi_2$  maps the function  $f(\lambda, \mu) = g(\lambda)h(\mu)$  to the transformator  $\varphi_1(g) \boxtimes \varphi_2(h)$ , where  $\varphi_1$  and  $\varphi_2$  are scalar functional calculi (Theorem 25) generated by the pseudo-resolvents  $R_{1,(\cdot)}$  and  $R_{2,(\cdot)}$ .

*Proof.* The proof is analogous to that of Theorem 34, see below.  $\square$

**Theorem 33.** *Assume that  $\infty \notin \bar{\sigma}(R_{1,(\cdot)})$ , but  $\infty \in \bar{\sigma}(R_{2,(\cdot)})$ . We define the mapping  $\varphi_1 \boxtimes \varphi_2: \mathbf{O}(\sigma(R_{1,(\cdot)}) \times \bar{\sigma}(R_{2,(\cdot)})) \rightarrow \mathbf{B}(X \boxtimes Y)$  by the formula*

$$(\varphi_1 \boxtimes \varphi_2)f = \frac{1}{(2\pi i)^2} \int_{\Gamma_1} \int_{\Gamma_2} f(\lambda, \mu) R_{1,\lambda} \boxtimes R_{2,\mu} d\mu d\lambda + \frac{1}{2\pi i} \int_{\Gamma_1} f(\lambda, \infty) R_{1,\lambda} \boxtimes \mathbf{1}_Y d\lambda,$$

where  $\Gamma_1$  is an oriented envelope of the singular set  $\sigma(R_{1,(\cdot)})$  with respect to the point  $\infty$  and the complement  $\bar{\mathbb{C}} \setminus U_1$ , and  $\Gamma_2$  is an oriented envelope of the extended singular set  $\bar{\sigma}(R_{2,(\cdot)})$  with respect to the complement  $\bar{\mathbb{C}} \setminus U_2$ ; here  $U_1 \times U_2$  is an open neighbourhood of the set  $\sigma(R_{1,(\cdot)}) \times \bar{\sigma}(R_{2,(\cdot)})$  that lies in the domain of the function  $f$  (see Proposition 9). We assert that  $\varphi_1 \boxtimes \varphi_2$  is a continuous morphism of unital algebras.

For all  $g \in \mathbf{O}(\sigma(R_{1,(\cdot)}))$  and  $h \in \mathbf{O}(\bar{\sigma}(R_{2,(\cdot)}))$  the morphism  $\varphi_1 \boxtimes \varphi_2$  maps the function  $f(\lambda, \mu) = g(\lambda)h(\mu)$  to the transformator  $\varphi_1(g) \boxtimes \varphi_2(h)$ , where  $\varphi_1$  and  $\varphi_2$  are scalar functional calculi (Theorems 25 and 26) generated by the pseudo-resolvents  $R_{1,(\cdot)}$  and  $R_{2,(\cdot)}$ .

*Proof.* The proof is analogous to that of Theorem 34, see below.  $\square$



**Theorem 34.** Assume that  $\infty \in \bar{\sigma}(R_{1,(\cdot)})$  and  $\infty \in \bar{\sigma}(R_{2,(\cdot)})$ . We define the mapping  $\varphi_1 \boxtimes \varphi_2: \mathbf{O}(\bar{\sigma}(R_{1,(\cdot)}) \times \bar{\sigma}(R_{2,(\cdot)})) \rightarrow \mathbf{B}(X \boxtimes Y)$  by the formula

$$\begin{aligned} (\varphi_1 \boxtimes \varphi_2)f &= \frac{1}{(2\pi i)^2} \int_{\Gamma_1} \int_{\Gamma_2} f(\lambda, \mu) R_{1,\lambda} \boxtimes R_{2,\mu} d\mu d\lambda + \frac{1}{2\pi i} \int_{\Gamma_1} f(\lambda, \infty) R_{1,\lambda} \boxtimes \mathbf{1} d\lambda \\ &\quad + \frac{1}{2\pi i} \int_{\Gamma_2} f(\infty, \mu) \mathbf{1} \boxtimes R_{2,\mu} d\mu + f(\infty, \infty) \mathbf{1}_{X \boxtimes Y}, \end{aligned}$$

where  $\Gamma_1$  and  $\Gamma_2$  are oriented envelopes of the singular sets  $\bar{\sigma}(R_{1,(\cdot)})$  and  $\bar{\sigma}(R_{2,(\cdot)})$  with respect to the complements  $\bar{\mathbb{C}} \setminus U_1$  and  $\bar{\mathbb{C}} \setminus U_2$ ; here  $U_1 \times U_2$  is an open neighbourhood of the set  $\bar{\sigma}(R_{1,(\cdot)}) \times \bar{\sigma}(R_{2,(\cdot)})$  that lies in the domain of the function  $f$  (see Proposition 9). We assert that  $\varphi_1 \boxtimes \varphi_2$  is a continuous morphism of unital algebras.

For all  $g \in \mathbf{O}(\bar{\sigma}(R_{1,(\cdot)}))$  and  $h \in \mathbf{O}(\bar{\sigma}(R_{2,(\cdot)}))$  the morphism  $\varphi_1 \boxtimes \varphi_2$  maps the function  $f(\lambda, \mu) = g(\lambda)h(\mu)$  to the transformator  $\varphi_1(g) \boxtimes \varphi_2(h)$ , where  $\varphi_1$  and  $\varphi_2$  are scalar functional calculi (Theorem 26) generated by the pseudo-resolvents  $R_{1,(\cdot)}$  and  $R_{2,(\cdot)}$ .

*Proof.* For each  $\mu \in U_2$  we consider the operator

$$G(\mu) = \varphi_2(f(\cdot, \mu)) = \frac{1}{2\pi i} \int_{\Gamma_1} f(\lambda, \mu) R_{1,\lambda} d\lambda + f(\infty, \mu) \mathbf{1}_X. \quad (18)$$

By Theorem 26, for any fixed  $\mu \in U_2$  the correspondence  $f \mapsto G(\mu)$  preserves the three operations: addition, scalar multiplication, and multiplication. We change the interpretation: formula (18) defines a mapping  $f \mapsto G$  from  $\mathbf{O}(\bar{\sigma}(R_{1,(\cdot)}) \times \bar{\sigma}(R_{2,(\cdot)}))$  to  $\mathbf{O}(\bar{\sigma}(R_{2,(\cdot)}), \mathbf{B}_0(X))$ . Since the three operations in  $\mathbf{O}(\bar{\sigma}(R_{2,(\cdot)}), \mathbf{B}_0(X))$  are understood in the pointwise sense, it follows that the correspondence  $f \mapsto G$  is a morphism of algebras.

In accordance with Theorem 29 we put

$$\begin{aligned} \Phi_1(G) &= \frac{1}{2\pi i} \int_{\Gamma_2} G(\mu) \boxtimes R_{2,\mu} d\mu + G(\infty) \boxtimes \mathbf{1}_Y \\ &= \frac{1}{2\pi i} \int_{\Gamma_2} \left( \frac{1}{2\pi i} \int_{\Gamma_1} f(\lambda, \mu) R_{1,\lambda} d\lambda + f(\infty, \mu) \mathbf{1}_X \right) \boxtimes R_{2,\mu} d\mu \\ &\quad + \left( \frac{1}{2\pi i} \int_{\Gamma_1} f(\lambda, \infty) R_{1,\lambda} d\lambda + f(\infty, \infty) \mathbf{1}_X \right) \boxtimes \mathbf{1}_Y \\ &= \frac{1}{2\pi i} \int_{\Gamma_2} \left( \frac{1}{2\pi i} \int_{\Gamma_1} f(\lambda, \mu) R_{1,\lambda} d\lambda \right) \boxtimes R_{2,\mu} d\mu \\ &\quad + \frac{1}{2\pi i} \int_{\Gamma_2} \left( f(\infty, \mu) \mathbf{1}_X \right) \boxtimes R_{2,\mu} d\mu \\ &\quad + \left( \frac{1}{2\pi i} \int_{\Gamma_1} f(\lambda, \infty) R_{1,\lambda} d\lambda + f(\infty, \infty) \mathbf{1}_X \right) \boxtimes \mathbf{1}_Y \\ &= \frac{1}{(2\pi i)^2} \int_{\Gamma_1} \int_{\Gamma_2} f(\lambda, \mu) R_{1,\lambda} \boxtimes R_{2,\mu} d\mu d\lambda + \frac{1}{2\pi i} \int_{\Gamma_2} f(\infty, \mu) \mathbf{1}_X \boxtimes R_{2,\mu} d\mu \\ &\quad + \frac{1}{2\pi i} \int_{\Gamma_1} f(\lambda, \infty) R_{1,\lambda} \boxtimes \mathbf{1}_Y d\lambda + f(\infty, \infty) \mathbf{1}_X \boxtimes \mathbf{1}_Y. \end{aligned} \quad (19)$$

By Theorem 29, the correspondence  $G \mapsto \Phi_1(G)$  also preserves the three operations. Clearly, the mapping  $\varphi_1 \boxtimes \varphi_2$  from the formulation of the theorem is the composition of the correspondences  $f \mapsto G$  and  $G \mapsto \Phi_1(G)$ , and, by what has been proved, is a morphism of algebras.

The continuity is evident.

The second statement is verified by direct calculations.  $\square$

When it is desirable to stress that in Theorems 32, 33, and 34, the functional calculus  $\varphi_1 \boxtimes \varphi_2$  is generated by pseudo-resolvents  $R_{1,(\cdot)}$  and  $R_{2,(\cdot)}$ , we will use the notation  $\varphi_{R_{1,(\cdot)}} \boxtimes \varphi_{R_{2,(\cdot)}}$  instead of  $\varphi_1 \boxtimes \varphi_2$ , and we will use the notation  $f(R_{1,(\cdot)}, R_{2,(\cdot)})$  instead of  $(\varphi_1 \boxtimes \varphi_2)(f)$ . If the pseudo-resolvents  $R_{1,(\cdot)}$  and  $R_{2,(\cdot)}$  are generated by the operators  $A$  and  $B$  (see Proposition 24), we will use the notations  $\varphi_A \boxtimes \varphi_B$  and  $f(A, B)$ .

In order to present the definitions of  $\varphi_1 \boxtimes \varphi_2$  from Theorems 32, 33, and 34 in a unified form, it is convenient to use the notation

$$\begin{aligned} (\varphi_1 \boxtimes \varphi_2)(f) &= \frac{1}{(2\pi i)^2} \int_{\Gamma_1} \int_{\Gamma_2} f(\lambda, \mu) R_{1,\lambda} \boxtimes R_{2,\mu} d\mu d\lambda \\ &\quad + \delta_2 \frac{1}{2\pi i} \int_{\Gamma_1} f(\lambda, \infty) R_{1,\lambda} \boxtimes \mathbf{1} d\lambda \\ &\quad + \delta_1 \frac{1}{2\pi i} \int_{\Gamma_2} f(\infty, \mu) \mathbf{1} \boxtimes R_{2,\mu} d\mu + \delta_1 \delta_2 f(\infty, \infty) \mathbf{1} \boxtimes \mathbf{1}, \end{aligned} \quad (20)$$

where  $\delta_i = 1$  if  $\Gamma_i$  encloses  $\infty$ , and  $\delta_i = 0$  in the opposite case,  $i = 1, 2$ .

We enumerate the results of the action of  $\varphi_1 \boxtimes \varphi_2$  on some frequently encountered functions.

**Corollary 35.** *Under the assumptions of Theorems 32, 33, and 34, the morphism  $\varphi_1 \boxtimes \varphi_2$  maps the function  $u(\lambda, \mu) = 1$  to the unit  $\mathbf{1} \boxtimes \mathbf{1}$  of the algebra  $\mathbf{B}(X \boxtimes Y)$ ; the function  $r_{1,\lambda_0}(\lambda, \mu) = \frac{1}{\lambda_0 - \lambda}$ , where  $\lambda_0 \in \rho(R_{1,(\cdot)})$ , is mapped by the morphism  $\varphi_1 \boxtimes \varphi_2$  to the transformer  $R_{1,\lambda_0} \boxtimes \mathbf{1}_Y$ ; the function  $r_{2,\mu_0}(\lambda, \mu) = \frac{1}{\mu_0 - \mu}$ , where  $\mu_0 \in \rho(R_{2,(\cdot)})$ , is mapped by the morphism  $\varphi_1 \boxtimes \varphi_2$  to the transformer  $\mathbf{1}_X \boxtimes R_{2,\mu_0}$ ; the function  $r_{\lambda_0,\mu_0}(\lambda, \mu) = \frac{1}{(\lambda_0 - \lambda)(\mu_0 - \mu)}$ , where  $\lambda_0 \in \rho(R_{1,(\cdot)})$  and  $\mu_0 \in \rho(R_{2,(\cdot)})$ , is mapped by the morphism  $\varphi_1 \boxtimes \varphi_2$  to the transformer  $R_{1,\lambda_0} \boxtimes R_{2,\mu_0}$ .*

*Under the assumptions of Theorems 32 and 33, the morphism  $\varphi_1 \boxtimes \varphi_2$  maps the function  $c_1(\lambda, \mu) = \lambda$  to the transformer  $A \boxtimes \mathbf{1}_Y$ , where  $A$  is the operator that generates the maximal pseudo-resolvent  $R_{1,(\cdot)}$  in accordance with Proposition 24.*

*Under the assumptions of Theorem 32, the morphism  $\varphi_1 \boxtimes \varphi_2$  maps the function  $c_2(\lambda, \mu) = \mu$  to the transformer  $\mathbf{1}_X \boxtimes B$ , where  $B$  is the operator that generates the maximal pseudo-resolvent  $R_{2,(\cdot)}$  in accordance with Proposition 24; the function  $r_{\nu_0}(\lambda, \mu) = \frac{1}{\nu_0 - \lambda \mp \mu}$  is mapped by the morphism  $\varphi_1 \boxtimes \varphi_2$  to the transformer  $(\nu_0 \mathbf{1} \boxtimes \mathbf{1} - A \boxtimes \mathbf{1} \mp \mathbf{1} \boxtimes B)^{-1}$  provided  $\nu_0 \notin \sigma(A) \pm \sigma(B)$ .*

*Proof.* We restrict ourselves to proving the last statement. Clearly, the function  $(\lambda, \mu) \mapsto \nu_0 - \lambda \mp \mu$  is mapped by the morphism  $\varphi_1 \boxtimes \varphi_2$  to the transformer  $\nu_0 \mathbf{1} \boxtimes \mathbf{1} - A \boxtimes \mathbf{1} \mp \mathbf{1} \boxtimes B$ . Since  $\varphi_1 \boxtimes \varphi_2$  is a morphism of algebras, the reciprocal function is mapped to the inverse transformer.  $\square$

**Theorem 36.** *Let  $g \in \mathbf{O}(\bar{\sigma}(R_{1,(\cdot)}) \times \bar{\sigma}(R_{2,(\cdot)}))$  and  $f \in \mathbf{O}(g(\bar{\sigma}(R_{1,(\cdot)}) \times \bar{\sigma}(R_{2,(\cdot)})))$ . Then the transformer  $(\varphi_1 \boxtimes \varphi_2)(f \circ g)$  is the function  $f$  of the transformer  $(\varphi_1 \boxtimes \varphi_2)(g)$ :*

$$(\varphi_1 \boxtimes \varphi_2)(f \circ g) = \frac{1}{2\pi i} \int_{\Gamma_3} f(\nu) (\nu \mathbf{1} \boxtimes \mathbf{1} - (\varphi_1 \boxtimes \varphi_2)(g))^{-1} d\nu,$$

where  $\Gamma_3$  is an oriented envelope of the spectrum  $\sigma((\varphi_1 \boxtimes \varphi_2)(g))$ .

*Proof.* We have  $(\delta_1, \delta_2 = 0, 1)$

$$\begin{aligned}
(\varphi_1 \boxtimes \varphi_2)(f \circ g) &= \frac{1}{(2\pi i)^2} \int_{\Gamma_1} \int_{\Gamma_2} f(g(\lambda, \mu)) R_{1,\lambda} \boxtimes R_{2,\mu} d\mu d\lambda \\
&+ \delta_2 \frac{1}{2\pi i} \int_{\Gamma_1} f(g(\lambda, \infty)) R_{1,\lambda} \boxtimes \mathbf{1} d\lambda \\
&+ \delta_1 \frac{1}{2\pi i} \int_{\Gamma_2} f(g(\infty, \mu)) \mathbf{1} \boxtimes R_{2,\mu} d\mu + \delta_1 \delta_2 f(g(\infty, \infty)) \mathbf{1}_{X \boxtimes Y} \\
&= \frac{1}{(2\pi i)^2} \int_{\Gamma_1} \int_{\Gamma_2} \left[ \frac{1}{2\pi i} \int_{\Gamma_3} \frac{f(\nu)}{\nu - g(\lambda, \mu)} d\nu \right] R_{1,\lambda} \boxtimes R_{2,\mu} d\mu d\lambda \\
&+ \delta_2 \frac{1}{2\pi i} \int_{\Gamma_1} \left[ \frac{1}{2\pi i} \int_{\Gamma_3} \frac{f(\nu)}{\nu - g(\lambda, \infty)} d\nu \right] R_{1,\lambda} \boxtimes \mathbf{1} d\lambda \\
&+ \delta_1 \frac{1}{2\pi i} \int_{\Gamma_2} \left[ \frac{1}{2\pi i} \int_{\Gamma_3} \frac{f(\nu)}{\nu - g(\infty, \mu)} d\nu \right] \mathbf{1} \boxtimes R_{2,\mu} d\mu + \delta_1 \delta_2 f(g(\infty, \infty)) \mathbf{1}_{X \boxtimes Y}
\end{aligned}$$

(here we interchange the order of integration)

$$\begin{aligned}
&= \frac{1}{2\pi i} \int_{\Gamma_3} f(\nu) \left[ \frac{1}{(2\pi i)^2} \int_{\Gamma_1} \int_{\Gamma_2} \frac{R_{1,\lambda} \boxtimes R_{2,\mu}}{\nu - g(\lambda, \mu)} d\mu d\lambda \right] d\nu \\
&+ \delta_2 \frac{1}{2\pi i} \int_{\Gamma_3} f(\nu) \left[ \frac{1}{2\pi i} \int_{\Gamma_1} \frac{R_{1,\lambda} \boxtimes \mathbf{1}}{\nu - g(\lambda, \infty)} d\lambda \right] d\nu \\
&+ \delta_1 \frac{1}{2\pi i} \int_{\Gamma_3} f(\nu) \left[ \frac{1}{2\pi i} \int_{\Gamma_2} \frac{\mathbf{1} \boxtimes R_{2,\mu}}{\nu - g(\infty, \mu)} d\mu \right] d\nu + \delta_1 \delta_2 f(g(\infty, \infty)) \mathbf{1}_{X \boxtimes Y}
\end{aligned}$$

(further, by Theorems 32, 33, and 34, it follows that)

$$\begin{aligned}
&= \frac{1}{2\pi i} \int_{\Gamma_3} f(\nu) \left[ (\nu \mathbf{1} \boxtimes \mathbf{1} - (\varphi_1 \boxtimes \varphi_2)(g))^{-1} - \delta_2 \frac{1}{2\pi i} \int_{\Gamma_1} \frac{R_{1,\lambda} \boxtimes \mathbf{1}}{\nu - g(\lambda, \infty)} d\lambda \right. \\
&- \delta_1 \frac{1}{2\pi i} \int_{\Gamma_2} \frac{\mathbf{1} \boxtimes R_{2,\mu}}{\nu - g(\infty, \mu)} d\mu - \delta_1 \delta_2 g(\infty, \infty) \mathbf{1}_{X \boxtimes Y} \left. \right] d\nu \\
&+ \delta_2 \frac{1}{2\pi i} \int_{\Gamma_3} f(\nu) \left[ \frac{1}{2\pi i} \int_{\Gamma_1} \frac{R_{1,\lambda} \boxtimes \mathbf{1}}{\nu - g(\lambda, \infty)} d\lambda \right] d\nu \\
&+ \delta_1 \frac{1}{2\pi i} \int_{\Gamma_3} f(\nu) \left[ \frac{1}{2\pi i} \int_{\Gamma_2} \frac{\mathbf{1} \boxtimes R_{2,\mu}}{\nu - g(\infty, \mu)} d\mu \right] d\nu + \delta_1 \delta_2 f(g(\infty, \infty)) \mathbf{1}_{X \boxtimes Y} \\
&= \frac{1}{2\pi i} \int_{\Gamma_3} f(\nu) (\nu \mathbf{1} \boxtimes \mathbf{1} - (\varphi_1 \boxtimes \varphi_2)(g))^{-1} d\nu. \quad \square
\end{aligned}$$

**Corollary 37.** Let  $A \in \mathbf{B}(X)$  and  $B \in \mathbf{B}(Y)$ . Let  $f \in \mathbf{O}(\sigma(A) \pm \sigma(B))$ . Then

$$\frac{1}{(2\pi i)^2} \int_{\Gamma_1} \int_{\Gamma_2} f(\lambda \pm \mu) R_{A,\lambda} \boxtimes R_{B,\mu} d\mu d\lambda = \frac{1}{2\pi i} \int_{\Gamma_3} f(\nu) (\nu \mathbf{1} \boxtimes \mathbf{1} - A \boxtimes \mathbf{1} \mp \mathbf{1} \boxtimes B)^{-1} d\nu,$$

where  $\Gamma_3$  is an oriented envelope of  $\sigma(A) \pm \sigma(B)$ .

*Proof.* This is a special case of Theorem 36 for  $g(\lambda, \mu) = \lambda \pm \mu$ .  $\square$

*Example 4.* Let  $A \in \mathbf{B}(X)$  and  $B \in \mathbf{B}(Y)$ . By Corollary 37 and the formula  $e^{\lambda t} e^{\mu t} = e^{(\lambda + \mu)t}$ , one has (cf. [12, 53], [65, Theorem 10.9])

$$e^{At} \boxtimes e^{Bt} = e^{(A \boxtimes \mathbf{1} + \mathbf{1} \boxtimes B)t}.$$

We proceed to the discussion of spectral mapping theorems.

**Theorem 38.** *Let  $f \in \mathbf{O}(\bar{\sigma}(R_{1,(\cdot)}) \times \bar{\sigma}(R_{2,(\cdot)}))$ . Then the transformator  $(\varphi_1 \boxtimes \varphi_2)(f): X \boxtimes Y \rightarrow X \boxtimes Y$  is not invertible if and only if  $f(\lambda, \mu) = 0$  for at least one couple of points  $\lambda \in \bar{\sigma}(R_{1,(\cdot)})$  and  $\mu \in \bar{\sigma}(R_{2,(\cdot)})$ .*

*Proof.* For each  $\mu \in U_2$ , we consider operator (18). By Theorem 27, the following statement holds: the operator  $G(\mu): X \rightarrow X$  is not invertible if and only if  $f(\lambda, \mu) = 0$  for at least one  $\lambda \in \bar{\sigma}(R_{1,(\cdot)})$ . Further, by Theorem 30, operator (19) is not invertible if and only if  $G(\mu)$  is not invertible for at least one  $\mu \in \bar{\sigma}(R_{2,(\cdot)})$ . Combining (in the opposite order) all these results, we arrive at the desired statement.  $\square$

**Theorem 39.** *Let  $f \in \mathbf{O}(\bar{\sigma}(R_{1,(\cdot)}) \times \bar{\sigma}(R_{2,(\cdot)}))$ . Then the spectrum of the transformator  $(\varphi_1 \boxtimes \varphi_2)f: X \boxtimes Y \rightarrow X \boxtimes Y$  is given by the formula*

$$\sigma((\varphi_1 \boxtimes \varphi_2)f) = \{ f(\lambda, \mu) : \lambda \in \bar{\sigma}(R_{1,(\cdot)}), \mu \in \bar{\sigma}(R_{2,(\cdot)}) \}.$$

*Proof.* We take an arbitrary  $\nu \in \mathbb{C}$ . By the definition of the spectrum, the number  $\nu$  belongs to the set  $\sigma((\varphi_1 \boxtimes \varphi_2)f)$  if and only if the transformator  $\nu \mathbf{1}_{X \boxtimes Y} - (\varphi_1 \boxtimes \varphi_2)(f)$  is not invertible.

We denote by  $u$  the function from  $\mathbf{O}(\bar{\sigma}(R_{1,(\cdot)}) \times \bar{\sigma}(R_{2,(\cdot)}))$  that identically equals 1. By Theorems 32, 33, and 34, we have

$$(\varphi_1 \boxtimes \varphi_2)(u) = \mathbf{1}_{X \boxtimes Y},$$

whence

$$\nu \mathbf{1}_{X \boxtimes Y} - (\varphi_1 \boxtimes \varphi_2)(f) = (\varphi_1 \boxtimes \varphi_2)(\nu u - f).$$

We apply Theorem 38: the transformator  $(\varphi_1 \boxtimes \varphi_2)(\nu u - f)$  is not invertible if and only if  $\nu u(\lambda, \mu) - f(\lambda, \mu) = 0$  for some  $\lambda \in \bar{\sigma}(R_{1,(\cdot)})$  and  $\mu \in \bar{\sigma}(R_{2,(\cdot)})$  or, in other words,  $\nu \in \{ f(\lambda, \mu) : \lambda \in \bar{\sigma}(R_{1,(\cdot)}), \mu \in \bar{\sigma}(R_{2,(\cdot)}) \}$ .  $\square$

We denote by  $\mathbf{B}_{R_1, R_2}$  the closure in  $\mathbf{B}(\mathbf{B}(X, Y))$  of the set of all transformators  $(\varphi_1 \boxtimes \varphi_2)f$ , where  $f \in \mathbf{O}(\bar{\sigma}(R_{1,(\cdot)}) \times \bar{\sigma}(R_{2,(\cdot)}))$ .

**Corollary 40.** *The set  $\mathbf{B}_{R_1, R_2}$  is a full commutative subalgebra of the algebra  $\mathbf{B}(\mathbf{B}(X, Y))$  of all transformators acting in  $\mathbf{B}(X, Y)$ .*

*Proof.* Clearly, the image under  $\varphi_1 \boxtimes \varphi_2$  of the unital commutative algebra  $\mathbf{O}(\bar{\sigma}(R_{1,(\cdot)}) \times \bar{\sigma}(R_{2,(\cdot)}))$  is a unital commutative subalgebra.

Let the transformator  $(\varphi_1 \boxtimes \varphi_2)f$  be invertible. By Theorem 39, this means that  $f(\lambda, \mu) \neq 0$  for some  $\lambda \in \bar{\sigma}(R_{1,(\cdot)})$  and  $\mu \in \bar{\sigma}(R_{2,(\cdot)})$ . Clearly, the inverse of  $(\varphi_1 \boxtimes \varphi_2)f$  is the transformator  $(\varphi_1 \boxtimes \varphi_2)\frac{1}{f}$ .

It remains to apply Proposition 4.  $\square$

In [113], an analogue of Theorem 32 was proved in the tensor product of Banach spaces for bounded operators and a polynomial function  $f$ . In [69, Theorem 2.4], an analogue of Theorem 32 was proved in the tensor product of Banach spaces for bounded operators and an arbitrary analytic function  $f$ ; in [69, Theorem 2.4], an analogue of Theorem 34 was also proved for unbounded operators and analytic functions. See also the initial version [68] of article [69].

An analogue of Theorem 36 for matrices was proved in [85, Theorem 4.4].

There are several versions of Theorem 39 in tensor products of Banach spaces. It was shown in [19] that the spectrum of the tensor product  $A \otimes B$  of two bounded operators acting in a Hilbert space is the set  $\sigma(A) \times \sigma(B)$ . For functions  $f$  of the form  $f(\lambda, \mu) =$

$g(\lambda)h(\mu)$ , Theorem 39 was proved in [95]; for polynomial functions  $f$  of two variables, a version of Theorem 39 was proved in [60, Theorem 3.3], see also [59]; another equivalent version was proved in [37, Theorem 3.4]. In [69, Theorem 3.2], it was proved an analogue of Theorem 39 for unbounded operators and analytic functions, see also [68]. A modern version of Theorem 39 for matrices can be found in [85, Lemma 4.1].

A functional calculus for the transformator  $A \otimes \mathbf{1} - \mathbf{1} \otimes B$  was first described in [109]. Functions of the transformator  $A \otimes \mathbf{1} \pm \mathbf{1} \otimes B$  are also investigated in [11, 12, 48].

## 7. MEROMORPHIC FUNCTIONAL CALCULUS

A meromorphic function of a bounded operator is an unbounded operator or a linear relation (provided a pole of the function is contained in the spectrum). According to our approach, we identify such an object with its resolvent.

Let  $U$  be an open subset of  $\overline{\mathbb{C}}^2$  and  $f : U \rightarrow \overline{\mathbb{C}}$ . The function  $f$  is called [115, ch. IV, § 15.43] *meromorphic* if: (i)  $f$  is analytic on a set  $U \setminus M$ , where  $M$  is a nowhere dense closed subset of  $U$ , (ii)  $f$  cannot be analytically continued to any point of  $M$ , (iii) for any point  $\zeta \in M$  there exist a connected neighborhood  $V$  of  $\zeta$  and an analytic function  $q_\zeta : V \rightarrow \mathbb{C}$  such that the function  $p_\zeta = f \cdot q_\zeta$  is analytic in  $V \cap (U \setminus M)$  and can be extended analytically into  $V$ , and  $q_\zeta$  equals zero only on  $V \cap M$ . Clearly,  $q_\zeta(\zeta) = 0$ . The set  $M$  is called the *polar set of the function*  $f$ . It consists of points of two types: if  $p_\zeta(\zeta) \neq 0$  (and so  $\lim_{z \rightarrow \zeta} f(z) = \infty$ ), then  $\zeta$  is called a *pole*; if  $p_\zeta(\zeta) = 0$ , then  $\zeta$  is called a *point of indeterminacy*. In any neighbourhood of a point of indeterminacy, the function  $f$  takes any value from  $\mathbb{C}$  [115]. For example, for the function  $f(\lambda, \mu) = \lambda\mu$ , the points of indeterminacy are  $(0, \infty)$  and  $(\infty, 0)$ , for the function  $f(\lambda, \mu) = \frac{\lambda}{\mu}$ , the points of indeterminacy are  $(0, 0)$  and  $(\infty, \infty)$ , and for the function  $f(\lambda, \mu) = \lambda - \mu$ , the point of indeterminacy is  $(\infty, \infty)$ .

Assume that we are given an extended tensor product  $X \boxtimes Y$  of Banach spaces  $X$  and  $Y$ , and we are given pseudo-resolvents  $R_{1,(\cdot)}$  and  $R_{2,(\cdot)}$  in the algebras  $\mathbf{B}_0(X)$  and  $\mathbf{B}_0(Y)$  respectively.

We consider a function  $f$  that is meromorphic in a neighbourhood  $U \subseteq \overline{\mathbb{C}}^2$  of the set  $\bar{\sigma}(R_{1,(\cdot)}) \times \bar{\sigma}(R_{2,(\cdot)})$  and has no points of indeterminacy in  $U$ . We consider the subset

$$f(\bar{\sigma}(R_{1,(\cdot)}), \bar{\sigma}(R_{2,(\cdot)})) = \{ f(\lambda, \mu) : (\lambda, \mu) \in \bar{\sigma}(R_{1,(\cdot)}) \times \bar{\sigma}(R_{2,(\cdot)}) \}$$

of the set  $\overline{\mathbb{C}}$ . The set  $f(\bar{\sigma}(R_{1,(\cdot)}), \bar{\sigma}(R_{2,(\cdot)}))$  is compact, being the image under the continuous function  $f$  of the compact set  $\bar{\sigma}(R_{1,(\cdot)}) \times \bar{\sigma}(R_{2,(\cdot)})$ .

**Lemma 41.** *For any  $\nu \notin f(\bar{\sigma}(R_{1,(\cdot)}), \bar{\sigma}(R_{2,(\cdot)}))$  the set*

$$(\overline{\mathbb{C}}^2 \setminus U) \cup \{ (\lambda, \mu) \in U : f(\lambda, \mu) = \nu \} \quad (21)$$

*is closed in  $\overline{\mathbb{C}}^2$  and does not intersect  $\bar{\sigma}(R_{1,(\cdot)}) \times \bar{\sigma}(R_{2,(\cdot)})$ . Moreover, for any closed set  $W \subseteq \overline{\mathbb{C}} \setminus f(\bar{\sigma}(R_{1,(\cdot)}), \bar{\sigma}(R_{2,(\cdot)}))$ , the set*

$$(\overline{\mathbb{C}}^2 \setminus U) \cup \{ (\lambda, \mu) \in U : f(\lambda, \mu) \in W \} \quad (22)$$

*is closed in  $\overline{\mathbb{C}}^2$  and does not intersect  $\bar{\sigma}(R_{1,(\cdot)}) \times \bar{\sigma}(R_{2,(\cdot)})$ .*

*Proof.* The set  $\overline{\mathbb{C}}^2 \setminus U$  is closed, being a complement of an open one. The set  $\{ (\lambda, \mu) \in U : f(\lambda, \mu) \in W \} = f^{-1}(W)$  is closed in  $U$ , being the inverse image of the closed set  $W$  under the continuous function  $f$ . This means that limit points of the set  $f^{-1}(W) =$

$\{(\lambda, \mu) \in U : f(\lambda, \mu) \in W\}$  either belongs to  $f^{-1}(W)$  or to the complement of  $\overline{\mathbb{C}}^2 \setminus U$ . Thus, set (22) is closed.

We show that set (22) is disjoint from  $\bar{\sigma}(R_{1,(\cdot)}) \times \bar{\sigma}(R_{2,(\cdot)})$ . Actually, if  $f(\lambda, \mu) = \nu \in W$  and  $(\lambda, \mu) \in \bar{\sigma}(R_{1,(\cdot)}) \times \bar{\sigma}(R_{2,(\cdot)})$ , then  $\nu \in f(\bar{\sigma}(R_{1,(\cdot)}), \bar{\sigma}(R_{2,(\cdot)}))$ , which contradicts the assumption. If  $(\lambda, \mu) \notin U$ , then  $(\lambda, \mu) \notin \bar{\sigma}(R_{1,(\cdot)}) \times \bar{\sigma}(R_{2,(\cdot)})$  by the definition of  $U$ .  $\square$

For all  $\nu \in \mathbb{C} \setminus f(\bar{\sigma}(R_{1,(\cdot)}), \bar{\sigma}(R_{2,(\cdot)}))$ , we set

$$\begin{aligned} S_\nu &= \frac{1}{(2\pi i)^2} \int_{\Gamma_1} \int_{\Gamma_2} \frac{1}{\nu - f(\lambda, \mu)} R_{1,\lambda} \boxtimes R_{2,\mu} d\mu d\lambda \\ &+ \delta_2 \frac{1}{2\pi i} \int_{\Gamma_1} \frac{1}{\nu - f(\lambda, \infty)} R_{1,\lambda} \boxtimes \mathbf{1} d\lambda \\ &+ \delta_1 \frac{1}{2\pi i} \int_{\Gamma_2} \frac{1}{\nu - f(\infty, \mu)} \mathbf{1} \boxtimes R_{2,\mu} d\mu + \frac{\delta_1 \delta_2}{\nu - f(\infty, \infty)} \mathbf{1} \boxtimes \mathbf{1}, \end{aligned} \quad (23)$$

where  $\Gamma_i$  is an oriented envelope of the spectrum  $\sigma(R_{i,(\cdot)})$ ;  $\delta_i = 1$  if  $\Gamma_i$  encloses  $\infty$ , and  $\delta_i = 0$  in the opposite case;  $i = 1, 2$ . By Lemma 41, the function  $h_\nu(\lambda, \mu) = \frac{1}{\nu - f(\lambda, \mu)}$  belongs to  $\mathbf{O}(\bar{\sigma}(R_{1,(\cdot)}) \times \bar{\sigma}(R_{2,(\cdot)}))$ . Therefore  $S_\nu$  can be regarded as the image under the morphism  $\varphi_1 \boxtimes \varphi_2$  of the function  $h_\nu$ :

$$S_\nu = (\varphi_1 \boxtimes \varphi_2) h_\nu.$$

We denote by  $S_\nu(R_{1,(\cdot)}, R_{2,(\cdot)})$  transformator (23) generated by the pseudo-resolvents  $R_{1,(\cdot)}$  and  $R_{2,(\cdot)}$ , and we call  $S_\nu$  the *resolvent of the function  $f$  of  $R_{1,\lambda}$  and  $R_{2,\lambda}$* .

**Theorem 42.** *Let the function  $f$  be meromorphic in an open neighbourhood  $U \subseteq \overline{\mathbb{C}}^2$  of the set  $\bar{\sigma}(R_{1,(\cdot)}) \times \bar{\sigma}(R_{2,(\cdot)})$  and have no points of indeterminacy in  $U$ . Then the family*

$$S_\nu, \quad \nu \notin f(\bar{\sigma}(R_{1,(\cdot)}), \bar{\sigma}(R_{2,(\cdot)})),$$

*defined by formula (23) is a maximal pseudo-resolvent. In particular,*

$$\bar{\sigma}(S_{(\cdot)}) = f(\bar{\sigma}(R_{1,(\cdot)}), \bar{\sigma}(R_{2,(\cdot)})). \quad (24)$$

Equality (24) can be considered as an analogue of the spectral mapping theorem.

*Proof.* We show that, on the set  $\mathbb{C} \setminus f(\bar{\sigma}(R_{1,(\cdot)}), \bar{\sigma}(R_{2,(\cdot)}))$ , the Hilbert identity holds:

$$S_{\nu_1} - S_{\nu_2} = -(\nu_1 - \nu_2) S_{\nu_1} S_{\nu_2}, \quad \nu_1, \nu_2 \notin f(\bar{\sigma}(R_{1,(\cdot)}), \bar{\sigma}(R_{2,(\cdot)})). \quad (25)$$

We note that

$$\frac{1}{\nu_1 - f(\lambda, \mu)} - \frac{1}{\nu_2 - f(\lambda, \mu)} = -\frac{\nu_1 - \nu_2}{(\nu_1 - f(\lambda, \mu))(\nu_2 - f(\lambda, \mu))}.$$

Applying the morphism  $\varphi_1 \boxtimes \varphi_2$  to this identity we arrive at the Hilbert identity (25).

We verify that the pseudo-resolvent  $S_{(\cdot)}$  is maximal. The validity of the Hilbert identity implies that

$$\sigma(S_{(\cdot)}) \subseteq f(\bar{\sigma}(R_{1,(\cdot)}), \bar{\sigma}(R_{2,(\cdot)})).$$

To prove the reverse inclusion, we fix an auxiliary point  $\nu \in \mathbb{C} \setminus f(\bar{\sigma}(R_{1,(\cdot)}), \bar{\sigma}(R_{2,(\cdot)}))$ . By Theorem 16, the pseudo-resolvent  $S_{(\cdot)}$  can be extended to points  $\eta \in \mathbb{C}$  in which the transformator  $\mathbf{1} + (\eta - \nu) S_\nu$  is invertible. By Theorems 32, 33, 34, and 39, and formula (23) we have

$$\sigma(S_\nu) = \left\{ \frac{1}{\nu - f(\lambda, \mu)} : (\lambda, \mu) \in \bar{\sigma}(R_{1,(\cdot)}) \times \bar{\sigma}(R_{2,(\cdot)}) \right\}.$$

It follows that

$$\begin{aligned}\sigma(\mathbf{1} + (\eta - \nu)S_\nu) &= \left\{ 1 + \frac{\eta - \nu}{\nu - f(\lambda, \mu)} : (\lambda, \mu) \in \bar{\sigma}(R_{1,(\cdot)}) \times \bar{\sigma}(R_{2,(\cdot)}) \right\} \\ &= \left\{ \frac{\eta - f(\lambda, \mu)}{\nu - f(\lambda, \mu)} : (\lambda, \mu) \in \bar{\sigma}(R_{1,(\cdot)}) \times \bar{\sigma}(R_{2,(\cdot)}) \right\}.\end{aligned}$$

From this formula it is seen that the transformator  $\mathbf{1} + (\eta - \nu)S_\nu$  is invertible if and only if

$$0 \notin \left\{ \frac{\eta - f(\lambda, \mu)}{\nu - f(\lambda, \mu)} : (\lambda, \mu) \in \bar{\sigma}(R_{1,(\cdot)}) \times \bar{\sigma}(R_{2,(\cdot)}) \right\},$$

which is equivalent to

$$\eta \notin \left\{ f(\lambda, \mu) : (\lambda, \mu) \in \bar{\sigma}(R_{1,(\cdot)}) \times \bar{\sigma}(R_{2,(\cdot)}) \right\}.$$

Thus, from  $\eta \notin f(\bar{\sigma}(R_{1,(\cdot)}), \bar{\sigma}(R_{2,(\cdot)}))$  it follows that  $\eta \notin \sigma(S_{(\cdot)})$ .

It remains to analyze the case  $\nu = \infty$ .

We assume that  $\infty \notin f(\bar{\sigma}(R_{1,(\cdot)}), \bar{\sigma}(R_{2,(\cdot)}))$ . Then, since the set  $f(\bar{\sigma}(R_{1,(\cdot)}), \bar{\sigma}(R_{2,(\cdot)}))$  is closed, a neighbourhood  $W \subseteq \mathbb{C}$  of infinity is also disjoint from  $f(\bar{\sigma}(R_{1,(\cdot)}), \bar{\sigma}(R_{2,(\cdot)}))$ . Without loss of generality, we may assume that the neighbourhood  $W$  is closed. By definition,  $S_\nu$  is defined for all  $\nu \in W \setminus \{\infty\}$ ; besides, by Lemma 41, we may assume that the contours  $\Gamma_1$  and  $\Gamma_2$  in (23) do not depend on  $\nu \in W \setminus \{\infty\}$ . We calculate the limit (see Theorems 32, 33, and 34):

$$\begin{aligned}\lim_{\nu \rightarrow \infty} \nu S_\nu &= \lim_{\nu \rightarrow \infty} \left( \frac{1}{(2\pi i)^2} \int_{\Gamma_1} \int_{\Gamma_2} \frac{\nu}{\nu - f(\lambda, \mu)} R_{1,\lambda} \boxtimes R_{2,\mu} d\mu d\lambda \right. \\ &\quad + \delta_2 \frac{1}{2\pi i} \int_{\Gamma_1} \frac{\nu}{\nu - f(\lambda, \infty)} R_{1,\lambda} \boxtimes \mathbf{1} d\lambda \\ &\quad \left. + \delta_1 \frac{1}{2\pi i} \int_{\Gamma_2} \frac{\nu}{\nu - f(\infty, \mu)} \mathbf{1} \boxtimes R_{2,\mu} d\mu + \frac{\delta_1 \delta_2 \nu}{\nu - f(\infty, \infty)} \mathbf{1} \boxtimes \mathbf{1} \right) \\ &= \lim_{\nu \rightarrow \infty} (\varphi_1 \boxtimes \varphi_2) \left( \frac{\nu}{\nu - f(\cdot, \cdot)} \right) = \lim_{\nu \rightarrow \infty} (\varphi_1 \boxtimes \varphi_2) u = \mathbf{1} \boxtimes \mathbf{1},\end{aligned}$$

because the functions  $\frac{\nu}{\nu - f(\cdot, \cdot)}$  converge to  $u$  as  $\nu \rightarrow \infty$  uniformly on  $\Gamma_1 \times \Gamma_2$ ; here  $u(\lambda, \mu) = 1$ . Consequently,  $\infty \notin \bar{\sigma}(S_{(\cdot)})$ .

Conversely, let  $\infty \notin \bar{\sigma}(S_{(\cdot)})$ . This means that  $S_{(\cdot)}$  is defined in a deleted neighbourhood  $W$  of infinity and

$$\begin{aligned}\lim_{\nu \rightarrow \infty} \nu S_\nu &= \lim_{\nu \rightarrow \infty} \left( \frac{1}{(2\pi i)^2} \int_{\Gamma_1} \int_{\Gamma_2} \frac{\nu}{\nu - f(\lambda, \mu)} R_{1,\lambda} \boxtimes R_{2,\mu} d\mu d\lambda \right. \\ &\quad + \delta_2 \frac{1}{2\pi i} \int_{\Gamma_1} \frac{\nu}{\nu - f(\lambda, \infty)} R_{1,\lambda} \boxtimes \mathbf{1} d\lambda \\ &\quad \left. + \delta_1 \frac{1}{2\pi i} \int_{\Gamma_2} \frac{\nu}{\nu - f(\infty, \mu)} \mathbf{1} \boxtimes R_{2,\mu} d\mu + \frac{\delta_1 \delta_2 \nu}{\nu - f(\infty, \infty)} \mathbf{1} \boxtimes \mathbf{1} \right) = \mathbf{1} \boxtimes \mathbf{1}.\end{aligned}\tag{26}$$

By the definition of  $S_{(\cdot)}$ , we have that  $W \cap f(\bar{\sigma}(R_{1,(\cdot)}), \bar{\sigma}(R_{2,(\cdot)})) = \emptyset$ . We show that  $\infty \notin f(\bar{\sigma}(R_{1,(\cdot)}), \bar{\sigma}(R_{2,(\cdot)}))$ .

Assuming the contrary, let  $f(\lambda_*, \mu_*) = \infty$  for a point  $(\lambda_*, \mu_*) \in \bar{\sigma}(R_{1,(\cdot)}) \times \bar{\sigma}(R_{2,(\cdot)})$ . Then, by Theorem 39,  $0 \in \bar{\sigma}(S_\nu)$  for any  $\nu \in W$ . We also have  $0 \in \bar{\sigma}(\nu S_\nu)$  for  $\nu \in W$ . By Corollaries 40 and 8, it follows that

$$0 \in \bar{\sigma}\left(\lim_{\nu \rightarrow \infty} \nu S_\nu\right),$$

which contradicts (26).  $\square$

Corollary 43, see below, answers in the affirmative to the question posed in [105, 107] about the independence of the definition of  $f(A, B)$  for unbounded operators  $A$  and  $B$  from the choice of sequences of bounded operators  $A_n$  and  $B_n$ , the resolvents of which converge to the resolvents of  $A$  and  $B$  respectively.

**Corollary 43.** *Let the sequences of pseudo-resolvents  $R_{n,1,(\cdot)}$  and  $R'_{n,1,(\cdot)}$  converge<sup>5</sup> to the same pseudo-resolvent  $R_{1,(\cdot)}$ , and let the sequences of pseudo-resolvents  $R_{n,2,(\cdot)}$  and  $R'_{n,2,(\cdot)}$  converge to the same pseudo-resolvent  $R_{2,(\cdot)}$ . Then both the sequence  $S_\nu(R_{n,1,(\cdot)}, R_{n,2,(\cdot)})$  and the sequence  $S_\nu(R'_{n,1,(\cdot)}, R'_{n,2,(\cdot)})$  converge to the pseudo-resolvent  $S_\nu(R_{1,(\cdot)}, R_{2,(\cdot)})$ .*

*Proof.* We make use of definition (23). By Lemma 22,  $R_{n,1,(\cdot)}$  and  $R'_{n,1,(\cdot)}$  converge to  $R_{1,(\cdot)}$  uniformly on  $\Gamma_1$ , and  $R_{n,2,(\cdot)}$  and  $R'_{n,2,(\cdot)}$  converge to  $R_{2,(\cdot)}$  uniformly on  $\Gamma_2$ . From formula (23) it is seen that  $S_\nu(R_{n,1,(\cdot)}, R_{n,2,(\cdot)})$  and  $S_\nu(R'_{n,1,(\cdot)}, R'_{n,2,(\cdot)})$  converge to  $S_\nu(R_{1,(\cdot)}, R_{2,(\cdot)})$ .  $\square$

The theory of meromorphic functions of one operator had its origin in the polynomial functional calculus for unbounded operators constructed in [124], see an exposition in [36, ch. VII, § 9]. Meromorphic functional calculus of one operator was constructed in [57]. A spectral mapping theorem for a polynomial of a linear relation was proved in [20, Theorem VI.5.4].

Polynomial functions of two unbounded operators were defined in [69, Theorem 3.4]; in particular, a spectral mapping theorem was established, see [69, Theorem 3.13]. Other analogues of the spectral mapping theorem for analytic functions of unbounded operators (including polynomials) were obtained in [107, Theorem 1] and [105, Theorem 4].

## 8. FUNCTIONAL CALCULUS $\varphi_1 \boxdot \varphi_2$

In this Section, we assume that we are given an extended tensor product  $X \boxtimes Y$  of Banach spaces  $X$  and  $Y$ , and we are given pseudo-resolvents  $R_{1,(\cdot)}$  and  $R_{2,(\cdot)}$  in the algebras  $\mathbf{B}_0(X)$  and  $\mathbf{B}_0(Y)$  respectively.

We define the mapping  $\varphi_1 \boxdot \varphi_2$  acting on functions  $f \in \mathbf{O}(\bar{\sigma}(R_{1,(\cdot)}) \cup \bar{\sigma}(R_{2,(\cdot)}))$  of one variable by the formula

$$(\varphi_1 \boxdot \varphi_2)f = \frac{1}{2\pi i} \int_{\Gamma} f(\lambda) R_{1,\lambda} \boxtimes R_{2,\lambda} d\lambda, \quad (27)$$

where  $\Gamma$  is an oriented envelope of the union  $\bar{\sigma}(R_{1,(\cdot)}) \cup \bar{\sigma}(R_{2,(\cdot)})$  of the extended singular sets with respect to the complement of the domain of  $f$ .

Let  $U \subseteq \overline{\mathbb{C}}$  be an open set and  $f : U \rightarrow \mathbb{C}$  be an analytic function. We call the *divided difference* [43, 73] of the function  $f$  the function  $f^{[1]} : U \times U \rightarrow \mathbb{C}$  defined by the formula

$$f^{[1]}(\lambda, \mu) = \begin{cases} \frac{f(\lambda) - f(\mu)}{\lambda - \mu}, & \text{if } \lambda \neq \mu, \\ f'(\lambda), & \text{if } \lambda = \mu, \\ 0, & \text{if } \lambda = \infty \text{ or } \mu = \infty. \end{cases} \quad (28)$$

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<sup>5</sup>See the definition on p. 10.



*Example 5.* We give examples of divided differences of some functions:

$$\begin{aligned}
v_1^{[1]}(\lambda, \mu) &= 1, & \text{where } v_1(\lambda) &= \lambda, \\
v_2^{[1]}(\lambda, \mu) &= \lambda + \mu, & \text{where } v_2(\lambda) &= \lambda^2, \\
v_n^{[1]}(\lambda, \mu) &= \lambda^{n-1} + \lambda^{n-2}\mu + \cdots + \mu^{n-1}, & \text{where } v_n(\lambda) &= \lambda^n, \\
v_{1/2}^{[1]}(\lambda, \mu) &= \frac{1}{\sqrt{\lambda} + \sqrt{\mu}}, & \text{where } v_{1/2}(\lambda) &= \sqrt{\lambda}, \\
r_1^{[1]}(\lambda, \mu) &= \frac{1}{(\lambda_0 - \lambda)(\lambda_0 - \mu)}, & \text{where } r_1(\lambda) &= \frac{1}{\lambda_0 - \lambda}, \\
r_n^{[1]}(\lambda, \mu) &= -\frac{\frac{1}{(\lambda_0 - \lambda)^n} - \frac{1}{(\lambda_0 - \mu)^n}}{(\lambda_0 - \lambda) - (\lambda_0 - \mu)} = \\
&= \frac{v_n^{[1]}(\lambda_0 - \lambda, \lambda_0 - \mu)}{(\lambda_0 - \lambda)^n(\lambda_0 - \mu)^n}, & \text{where } r_n(\lambda) &= \frac{1}{(\lambda_0 - \lambda)^n}.
\end{aligned}$$

The Taylor series for the divided difference of a function  $f$  at a point  $(\lambda_0, \lambda_0)$  has the form

$$f^{[1]}(\lambda, \mu) = \sum_{n=0}^{\infty} \frac{f^{(n+1)}(\lambda_0)}{(n+1)!} v_{n+1}^{[1]}(\lambda - \lambda_0, \mu - \lambda_0) = \sum_{n=0}^{\infty} \frac{f^{(n+1)}(\lambda_0)}{(n+1)!} \sum_{i=0}^n (\lambda - \lambda_0)^{n-i} (\mu - \lambda_0)^i,$$

where  $v_n(\lambda) = \lambda^n$ . In particular, for  $\exp_t(\lambda) = e^{\lambda t}$  and  $\exp_t^{(1)}(\lambda) = \lambda e^{\lambda t}$  we have

$$\begin{aligned}
\exp_t^{[1]}(\lambda, \mu) &= \sum_{n=0}^{\infty} \frac{t^n}{(n+1)!} v_{n+1}^{[1]}(\lambda, \mu) = \sum_{n=0}^{\infty} \frac{t^n}{(n+1)!} \sum_{i=0}^n \lambda^{n-i} \mu^i, \\
\exp_t^{(1)[1]}(\lambda, \mu) &= \sum_{n=0}^{\infty} \frac{t^n}{n!} v_{n+1}^{[1]}(\lambda, \mu) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{i=0}^n \lambda^{n-i} \mu^i.
\end{aligned}$$

**Proposition 44.** Let  $U \subseteq \overline{\mathbb{C}}$  be an open set and  $f : U \rightarrow \mathbb{C}$  be an analytic function. Then the function  $f^{[1]}$  is analytic in  $U \times U$ .

*Proof.* The analyticity at a finite point  $(\lambda, \mu)$ ,  $\lambda \neq \mu$ , is evident. The analyticity at the points of the form  $(\lambda, \infty)$  and  $(\infty, \mu)$ , where  $\lambda, \mu \in \mathbb{C}$ , is also evident.

We expand  $f$  in the Taylor series about a finite point  $\lambda_0 \neq \infty$ :

$$f(\lambda) = \sum_{n=0}^{\infty} c_n (\lambda - \lambda_0)^n.$$

It follows that for  $\lambda \neq \mu$  close to  $\lambda_0$  one has

$$f^{[1]}(\lambda, \mu) = \sum_{n=1}^{\infty} c_n v_n^{[1]}(\lambda - \lambda_0, \mu - \lambda_0),$$

where  $v_n^{[1]}(\lambda, \mu) = \lambda^{n-1} + \lambda^{n-2}\mu + \cdots + \mu^{n-1}$ . This series determines an analytic function in a neighbourhood of the point  $(\lambda_0, \lambda_0)$ . Clearly,  $f^{[1]}(\lambda_0, \lambda_0) = f'(\lambda_0)$ .

We expand  $f$  in the Laurent series with centre  $\infty$ :

$$f(\lambda) = \sum_{n=0}^{\infty} \frac{c_n}{\lambda^n}.$$

This formula shows that for  $\lambda \neq \mu$  close to  $\infty$  one has

$$f^{[1]}(\lambda, \mu) = - \sum_{n=1}^{\infty} c_n \frac{v_n^{[1]}(\lambda, \mu)}{\lambda^n \mu^n},$$

where  $v_n^{[1]}(\lambda, \mu) = \lambda^{n-1} + \lambda^{n-2}\mu + \dots + \mu^{n-1}$ . This series determines an analytic function in a neighbourhood of the point  $(\infty, \infty)$ . Clearly,  $f^{[1]}(\infty, \infty) = 0$ .  $\square$

**Theorem 45.** *Let  $f \in \mathbf{O}(\bar{\sigma}(R_{1,(\cdot)}) \cup \bar{\sigma}(R_{2,(\cdot)}))$ . Then<sup>6</sup>*

$$(\varphi_1 \boxdot \varphi_2)f = (\varphi_1 \boxtimes \varphi_2)f^{[1]}.$$

*The spectrum of the transformer  $(\varphi_1 \boxdot \varphi_2)f: X \boxtimes Y \rightarrow X \boxtimes Y$  is given by the formula*

$$\sigma((\varphi_1 \boxdot \varphi_2)f) = \{ f^{[1]}(\lambda, \mu) : \lambda \in \bar{\sigma}(R_{1,(\cdot)}), \mu \in \bar{\sigma}(R_{2,(\cdot)}) \}.$$

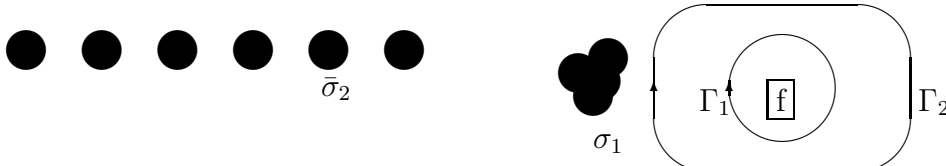


FIGURE 2. The contours  $\Gamma_1$  and  $\Gamma_2$  from the proof of Theorem 45. The localization of the complement of the domain of  $f$  is marked by  $\boxed{f}$

*Proof.* We take contours  $\Gamma_1$  and  $\Gamma_2$  such that the both are oriented envelopes of  $\bar{\sigma}(R_{1,(\cdot)}) \cup \bar{\sigma}(R_{2,(\cdot)})$  with respect to the complement of the domain of the function  $f$ , and  $\Gamma_2$  lies outside of  $\Gamma_1$  (so that  $\lambda - \mu$  does not vanish for  $\lambda \in \Gamma_1$  and  $\mu \in \Gamma_2$ ), see fig. 2. We make use of the definition:

$$\begin{aligned} (\varphi_1 \boxtimes \varphi_2)f^{[1]} &= \frac{1}{(2\pi i)^2} \int_{\Gamma_1} \int_{\Gamma_2} f^{[1]}(\lambda, \mu) R_{1,\lambda} \boxtimes R_{2,\mu} d\mu d\lambda \\ &+ \delta_2 \frac{1}{2\pi i} \int_{\Gamma_1} f^{[1]}(\lambda, \infty) R_{1,\lambda} \boxtimes \mathbf{1} d\lambda \\ &+ \delta_1 \frac{1}{2\pi i} \int_{\Gamma_2} f^{[1]}(\infty, \mu) \mathbf{1} \boxtimes R_{2,\mu} d\mu + \delta_1 \delta_2 f^{[1]}(\infty, \infty) \mathbf{1} \boxtimes \mathbf{1} \\ &= \frac{1}{(2\pi i)^2} \int_{\Gamma_1} \int_{\Gamma_2} \frac{f(\lambda) - f(\mu)}{\lambda - \mu} R_{1,\lambda} \boxtimes R_{2,\mu} d\mu d\lambda. \end{aligned}$$

We represent the last integral as the sum of two iterated integrals:

$$\frac{1}{2\pi i} \int_{\Gamma_1} R_{1,\lambda} \boxtimes \left( \frac{f(\lambda)}{2\pi i} \int_{\Gamma_2} \frac{1}{\lambda - \mu} R_{2,\mu} d\mu \right) d\lambda \quad (29)$$

$$+ \frac{1}{2\pi i} \int_{\Gamma_2} \left( \frac{f(\mu)}{2\pi i} \int_{\Gamma_1} \frac{1}{\mu - \lambda} R_{1,\lambda} d\lambda \right) \boxtimes R_{2,\mu} d\mu. \quad (30)$$

By the Cauchy integral formula, for the internal integral in (29) we have

$$\frac{1}{2\pi i} \int_{\Gamma_2} \frac{1}{\lambda - \mu} R_{2,\mu} d\mu = R_{2,\lambda},$$

<sup>6</sup>Strictly speaking, in this formula,  $f^{[1]}$  is understood to be the canonical projection of  $f^{[1]} \in \mathbf{O}[(\bar{\sigma}(R_{1,(\cdot)}) \cup \bar{\sigma}(R_{2,(\cdot)})) \times (\bar{\sigma}(R_{1,(\cdot)}) \cup \bar{\sigma}(R_{2,(\cdot)}))]$  into  $\mathbf{O}(\bar{\sigma}(R_{1,(\cdot)}) \times \bar{\sigma}(R_{2,(\cdot)}))$ .

and by the Cauchy integral theorem, for the internal integral in (30) we have

$$\frac{1}{2\pi i} \int_{\Gamma_1} \frac{1}{\mu - \lambda} R_{1,\lambda} d\lambda = 0,$$

since, by the assumption, the contour  $\Gamma_1$  does not surround the singularities of the function  $\lambda \mapsto \frac{1}{\mu - \lambda}$ ,  $\mu \in \Gamma_2$ , and the pseudo-resolvent  $\lambda \mapsto R_{1,\lambda}$ . Thus, the original integral takes the form

$$\frac{1}{2\pi i} \int_{\Gamma_1} f(\lambda) R_{1,\lambda} \boxtimes R_{2,\lambda} d\lambda = [(\varphi_1 \boxdot \varphi_2)f].$$

The second formula follows from 39. □

**Theorem 46.** *Let  $A \in \mathbf{B}(X)$ ,  $B \in \mathbf{B}(Y)$ , and  $f \in \mathbf{O}(\sigma(A) \cup \sigma(B))$ . Then*

$$\varphi_A(f) \boxtimes \mathbf{1} - \mathbf{1} \boxtimes \varphi_B(f) = [(\varphi_A \boxdot \varphi_B)f](A \boxtimes \mathbf{1} - \mathbf{1} \boxtimes B),$$

where the functional calculi  $\varphi_A$  and  $\varphi_B$  are constructed by  $A$  and  $B$  respectively.

*Proof.* The proof follows from the identity

$$f(\lambda) - f(\mu) = f^{[1]}(\lambda, \mu)(\lambda - \mu)$$

and Theorems 45 and 32. □

In the following corollary, we describe a representation for the increment of an analytic function.

**Corollary 47.** *Let  $A, B \in \mathbf{B}(X)$  and  $f \in \mathbf{O}(\sigma(A) \cup \sigma(B))$ . Then*

$$f(A) - f(B) = \frac{1}{2\pi i} \int_{\Gamma} f(\lambda)(\lambda \mathbf{1} - A)^{-1}(A - B)(\lambda \mathbf{1} - B)^{-1} d\lambda,$$

where the functional calculi  $\varphi_A$  and  $\varphi_B$  are constructed by  $A$  and  $B$  respectively, and  $\varphi_A \boxtimes \varphi_B$  acts in the extended tensor product  $\mathbf{B}(X, X)$ , see Example 3(e).

For the function  $f = \exp_t$ , this formula was found in [126, p. 978].

*Proof.* We apply the formula from Theorem 46 to the operator  $C = \mathbf{1}$ , assuming that  $X = Y$ . We have (taking into account that  $C = \mathbf{1}$ )

$$(A \boxtimes \mathbf{1} - \mathbf{1} \boxtimes B)C = AC - CB = A - B,$$

$$[(\varphi_A \boxdot \varphi_B)f](A - B) = \frac{1}{2\pi i} \int_{\Gamma} f(\lambda)(\lambda \mathbf{1} - A)^{-1}(A - B)(\lambda \mathbf{1} - B)^{-1} d\lambda,$$

$$(\varphi_A(f) \boxtimes \mathbf{1} - \mathbf{1} \boxtimes \varphi_B(f))C = \varphi_A(f)C - C\varphi_B(f) = f(A)C - Cf(B) = f(A) - f(B). \quad \square$$

One of the primary ideas [30, 40, 65, 88, 91, 99] of approximate calculation of an analytic function  $f$  of an operator or a pseudo-resolvent consists in an approximation of  $f$  by a polynomial or a rational function. In the case of  $(\varphi_1 \boxdot \varphi_2)f$ , for applying this idea it is necessary to be able to calculate  $\varphi_1 \boxdot \varphi_2$  at least of monomials and elementary rational functions. Formulae of this kind are presented in Corollary 48 below.

**Corollary 48.** *If the pseudo-resolvents  $R_{1,(\cdot)}$  and  $R_{2,(\cdot)}$  are generated by the operators  $A$  and  $B$  respectively, then*

$$(\varphi_A \boxdot \varphi_B)(v_n) = A^{n-1} \boxtimes \mathbf{1} + A^{n-2} \boxtimes B + \cdots + \mathbf{1} \boxtimes B^{n-1}, \quad \text{where } v_n(\lambda) = \lambda^n.$$

If  $\lambda_0 \in \rho(R_{1,(\cdot)}) \cap \rho(R_{2,(\cdot)})$ , then

$$(\varphi_1 \boxdot \varphi_2)(r_1) = R_{1,\lambda_0} \boxtimes R_{2,\lambda_0}, \quad \text{where } r_1(\lambda) = \frac{1}{\lambda_0 - \lambda}.$$

If, in addition, the extended singular sets of the pseudo-resolvents  $R_{1,(\cdot)}$  and  $R_{2,(\cdot)}$  are disjoint, then

$$(\varphi_1 \boxdot \varphi_2)(r_n) = - (R_{1,\lambda_0}^n \boxtimes \mathbf{1} - \mathbf{1} \boxtimes R_{2,\lambda_0}^n) (R_{1,\lambda_0} \boxtimes \mathbf{1} - \mathbf{1} \boxtimes R_{2,\lambda_0})^{-1}, \quad \text{where } r_n(\lambda) = \frac{1}{(\lambda_0 - \lambda)^n}.$$

If, in addition, the pseudo-resolvents  $R_{1,(\cdot)}$  and  $R_{2,(\cdot)}$  are generated by the operators  $A$  and  $B$  respectively, then

$$(\varphi_A \boxdot \varphi_B)(r_n) = ((\lambda_0 \mathbf{1} - A)^{n-1} \boxtimes \mathbf{1} + \cdots + \mathbf{1} \boxtimes (\lambda_0 \mathbf{1} - B)^{n-1}) (R_{A,\lambda_0}^n \boxtimes R_{B,\lambda_0}^n).$$

*Proof.* It suffices to make use of Example 5 and to apply Theorem 45 and Corollary 35.  $\square$

**Corollary 49.** Let  $g, h \in \mathbf{O}(\bar{\sigma}(R_{1,(\cdot)}) \cup \bar{\sigma}(R_{2,(\cdot)}))$ . Then

$$(\varphi_1 \boxdot \varphi_2)(gh) = [(\varphi_1 \boxdot \varphi_2)(g)](\mathbf{1} \boxtimes \varphi_2(h)) + (\varphi_1(g) \boxtimes \mathbf{1})[(\varphi_1 \boxdot \varphi_2)(h)],$$

where  $(gh)(\lambda) = g(\lambda)h(\lambda)$ .

*Proof.* By Theorem 45, this formula is equivalent to the identity

$$(\varphi_1 \boxtimes \varphi_2)(gh)^{[1]} = [(\varphi_1 \boxtimes \varphi_2)g^{[1]}](\mathbf{1} \boxtimes \varphi_2(h)) + (\varphi_1(g) \boxtimes \mathbf{1})[(\varphi_1 \boxtimes \varphi_2)h^{[1]}].$$

We have

$$\begin{aligned} \frac{g(\lambda)h(\lambda) - g(\mu)h(\mu)}{\lambda - \mu} &= \frac{g(\lambda)h(\lambda) - g(\mu)h(\lambda) + g(\mu)h(\lambda) - g(\mu)h(\mu)}{\lambda - \mu} \\ &= \frac{g(\lambda)h(\lambda) - g(\mu)h(\lambda)}{\lambda - \mu} + \frac{g(\mu)h(\lambda) - g(\mu)h(\mu)}{\lambda - \mu} \\ &= \frac{g(\lambda) - g(\mu)}{\lambda - \mu} h(\lambda) + g(\mu) \frac{h(\lambda) - h(\mu)}{\lambda - \mu}. \end{aligned}$$

Taking into account the ability of passages to the limits as  $\lambda - \mu \rightarrow 0$  and  $\lambda - \mu \rightarrow \infty$  we arrive at

$$(gh)^{[1]}(\lambda, \mu) = g(\lambda)h^{[1]}(\lambda, \mu) + g^{[1]}(\lambda, \mu)h(\mu).$$

It remains to apply Theorems 32, 33, and 34.  $\square$

The function  $\beta_{g,h} : U \times U \rightarrow \mathbb{C}$  defined by the formula

$$\beta_{g,h}(\lambda, \mu) = \begin{cases} \frac{g(\lambda)h(\mu) - h(\lambda)g(\mu)}{\lambda - \mu}, & \text{if } \lambda \neq \mu, \\ g'(\lambda)h(\mu) - h'(\lambda)g(\mu), & \text{if } \lambda = \mu, \\ 0, & \text{if } \lambda = \infty \text{ or } \mu = \infty, \end{cases}$$

is similar to the divided difference. It is generated by two analytic functions  $g, h : U \rightarrow \mathbb{C}$ . By analogy with [61, 62, 84], we call the function  $\beta_{g,h}$  the *Bezoutian*. The Bezoutian is a difference-differential analogue of the Wronskian. For example, the Bezoutian of the functions  $\sin$  and  $\cos$  is  $\text{sinc}(\lambda - \mu) = \frac{\sin(\lambda - \mu)}{\lambda - \mu}$ . We note that the Bezoutian can be expressed in terms of divided differences:

$$\beta_{g,h}(\lambda, \mu) = g^{[1]}(\lambda, \mu)h(\mu) - h^{[1]}(\lambda, \mu)g(\mu).$$

(In particular, this formula and Proposition 44 imply that  $\beta_{g,h}$  is an analytic function.) Conversely,

$$g^{[1]}(\lambda, \mu) = \beta_{g,u}(\lambda, \mu),$$

where  $u(\lambda) = 1$ .

**Corollary 50.** Let  $g, h \in \mathbf{O}(\bar{\sigma}(R_{1,(\cdot)}) \cup \bar{\sigma}(R_{2,(\cdot)}))$  and let  $h(\lambda) \neq 0$  for  $\lambda \in \bar{\sigma}(R_{1,(\cdot)}) \cup \bar{\sigma}(R_{2,(\cdot)})$ . Then

$$(\varphi_1 \boxdot \varphi_2) \left( \frac{g}{h} \right) = [(\varphi_1 \boxtimes \varphi_2)(\beta_{g,h})][\varphi_1(h) \boxtimes \varphi_2(h)]^{-1},$$

where  $\left( \frac{g}{h} \right)(\lambda) = \frac{g(\lambda)}{h(\lambda)}$ .

*Proof.* The proof is analogous to that of Corollary 49 and follows from the formula

$$\left[ \frac{g}{h} \right]^{[1]}(\lambda, \mu) = \frac{\beta_{g,h}(\lambda, \mu)}{h(\lambda)h(\mu)}. \quad \square$$

**Proposition 51.** Let  $X$  be a Banach space. We take for an extended tensor product the space  $\mathbf{B}(X, X)$  (see Example 3(e)), and we take for  $R_{1,(\cdot)}$  and  $R_{2,(\cdot)}$  the resolvent  $R_{(\cdot)}$  of the same operator  $A \in \mathbf{B}(X)$ . Let an operator  $C$  commute with at least one value  $R_\mu$  of the pseudo-resolvent  $R_{(\cdot)}$ . Then

$$[(\varphi_A \boxdot \varphi_A)f]C = f'(A)C = Cf'(A).$$

*Proof.* We note that, by virtue of Theorem 16,  $C$  commutes with all values  $R_\lambda$  of the pseudo-resolvent  $R_{(\cdot)}$ .

By the definition and commutativity, we have

$$[(\varphi \boxdot \varphi)f](C) = \frac{1}{2\pi i} \int_{\Gamma} f(\lambda) R_\lambda C R_\lambda d\lambda = \frac{1}{2\pi i} \int_{\Gamma} f(\lambda) R_\lambda^2 C d\lambda = \left( \frac{1}{2\pi i} \int_{\Gamma} f(\lambda) R_\lambda^2 d\lambda \right) C.$$

Passing to the limit in the Hilbert identity (3) we obtain the relation

$$R_\lambda^2 = -R'_\lambda, \quad \lambda \notin \sigma(R_{(\cdot)}).$$

Substituting this identity into the previous equality and integrating by parts we obtain

$$[(\varphi \boxdot \varphi)f](C) = -\left( \frac{1}{2\pi i} \int_{\Gamma} f(\lambda) R'_\lambda d\lambda \right) C = \left( \frac{1}{2\pi i} \int_{\Gamma} f'(\lambda) R_\lambda d\lambda \right) C = \varphi(f')C. \quad \square$$

We note that the divided differences  $f^{[1]}(A, B)$  of the operators  $A$  and  $B$  are also closely related to the calculation of functions of block triangular matrices [24, 25, 52, 65, 101].

## 9. THE IMPULSE RESPONSE

In subsequent sections, we discuss some applications.

In this Section, the previous results are applied to the representation of the impulse response of a second order differential equation. Here we regard the space  $\mathbf{B}(Y, X)$  (example 3(e)) as an extended tensor product. Therefore, for example, the action of the transformer  $\varphi_1(g) \boxtimes \varphi_2(h)$  on the operator  $C \in \mathbf{B}(Y, X)$  results in the operator  $\varphi_1(g)C\varphi_2(h)$ .

Let  $X$  and  $Y$  be Banach spaces and  $E, F, H \in \mathbf{B}(Y, X)$ . A function  $\lambda \mapsto \lambda^2 E + \lambda F + H$ , where  $\lambda \in \mathbb{C}$ , is called [42, 83, 96] a *square pencil*. The *resolvent set* of the pencil is the set  $\rho(E, F, H)$  of all  $\lambda \in \mathbb{C}$  such that the operator  $\lambda^2 E + \lambda F + H$  is invertible. The *spectrum* is the complement  $\sigma(E, F, H) = \mathbb{C} \setminus \rho(E, F, H)$  and the *resolvent* is the function

$$R_\lambda = (\lambda^2 E + \lambda F + H)^{-1}, \quad \lambda \in \rho(E, F, H). \quad (31)$$

The main sources [96, 125] of square pencils are the second order differential equations of the form

$$E\ddot{y}(t) + F\dot{y}(t) + Hy(t) = 0, \quad (32)$$

where  $y : \mathbb{R} \rightarrow Y$ . In this Section, it is always assumed that the operator  $E$  is invertible<sup>7</sup>. We recall the following proposition.

**Proposition 52** (see, for example, [93, Theorem 16]). *Let the operator  $E$  be invertible. Then the solution of the initial value problem*

$$\begin{aligned} E\ddot{y}(t) + F\dot{y}(t) + Hy(t) &= 0, \\ y(0) &= y_0, \\ \dot{y}(0) &= y_1 \end{aligned}$$

can be represented in the form

$$y(t) = \dot{\mathcal{T}}(t)Ey_0 + \mathcal{T}(t)(Ey_1 + Fy_0),$$

where

$$\begin{aligned} \mathcal{T}(t) &= \frac{1}{2\pi i} \int_{\Gamma} \exp_t(\lambda)(\lambda^2 E + \lambda F + H)^{-1} d\lambda, \\ \dot{\mathcal{T}}(t) &= \frac{1}{2\pi i} \int_{\Gamma} \exp_t^{(1)}(\lambda)(\lambda^2 E + \lambda F + H)^{-1} d\lambda, \end{aligned}$$

$\Gamma$  is an oriented envelope of the pencil spectrum  $\sigma(E, F, H)$ , and

$$\exp_t(\lambda) = e^{\lambda t}, \quad \exp_t^{(1)}(\lambda) = \lambda e^{\lambda t}.$$

It can be shown that the function  $\mathcal{T}$  is the impulse response, and  $\dot{\mathcal{T}}$  is its derivative.

A *factorization* of the pencil is the representation of its resolvent in the form

$$R_{\lambda} = R_{1,\lambda} C R_{2,\lambda}, \tag{33}$$

where  $R_{1,(\cdot)}$  and  $R_{2,(\cdot)}$  are pseudo-resolvents acting in  $X$  and  $Y$  respectively, and  $C \in \mathbf{B}(Y, X)$ . It is assumed that  $\rho(R_{1,(\cdot)}) \cap \rho(R_{2,(\cdot)}) \supseteq \rho(E, F, H)$ .

**Proposition 53.** *Let the operator  $E$  be invertible. Then we have  $C = E$  in formula (33), and the pseudo-resolvents  $R_{1,(\cdot)}$  and  $R_{2,(\cdot)}$  are the resolvents of some operators  $A_1$  and  $A_2$ .*

*Proof.* By Proposition 23, we have

$$\begin{aligned} R_{1,\lambda} &= -N_1 + \frac{P_1}{\lambda} + \frac{A_1}{\lambda^2} + \frac{A_1^2}{\lambda^3} + \frac{A_1^3}{\lambda^4} + \dots, \\ R_{2,\lambda} &= -N_2 + \frac{P_2}{\lambda} + \frac{A_2}{\lambda^2} + \frac{A_2^2}{\lambda^3} + \frac{A_2^3}{\lambda^4} + \dots \end{aligned}$$

Hence,

$$R_{1,\lambda} C R_{2,\lambda} = N_1 C N_2 - \frac{P_1 C N_2 + N_1 C P_2}{\lambda} + \frac{-A_1 C N_2 + P_1 C P_2 - N_1 C A_2}{\lambda^2} + \dots$$

On the other hand,

$$R_{\lambda} = \frac{E}{\lambda^2} - \frac{E^{-1} F E - 1}{\lambda^3} + \dots$$

Therefore,

$$N_1 C N_2 = \mathbf{0}, \quad P_1 C N_2 + N_1 C P_2 = \mathbf{0}, \quad -A_1 C N_2 + P_1 C P_2 - N_1 C A_2 = E.$$

---

<sup>7</sup>We note that even if  $E$  is invertible, the multiplication of the equation (32) by  $E^{-1}$  is not always desirable. For example, the operators  $E, F, H$  are often assumed [83, 114] to be self-adjoint, but the multiplication by  $E^{-1}$  may cause to the loss of this property.

Multiplying the second equation on the left by  $A_1P_1$  (keeping in mind the identities  $P^2 = P$ ,  $AP = PA = A$  and  $NP = PN = \mathbf{0}$  from Proposition 23), we arrive at

$$A_1CN_2 = \mathbf{0}.$$

Similarly, multiplying the second equation on the right by  $A_2$ , we have

$$N_1CA_2 = \mathbf{0}.$$

Substituting these results into the third equality, we obtain

$$P_1CP_2 = E.$$

Because of the invertibility of  $E$ , it follows that the projectors  $P_1$  and  $P_2$  coincide with  $\mathbf{1}$ , and  $C = E$ . Consequently (by the identity  $NP = PN = \mathbf{0}$ , see Proposition 23), we have  $N_1 = \mathbf{0}$  and  $N_2 = \mathbf{0}$ . It follows that  $\lim_{\lambda \rightarrow \infty} \lambda R_{1,\lambda} = \mathbf{1}$  and  $\lim_{\lambda \rightarrow \infty} \lambda R_{2,\lambda} = \mathbf{1}$ . By Proposition 24(c), this means that the pseudo-resolvents  $R_{1,(\cdot)}$  and  $R_{2,(\cdot)}$  are the resolvents of the operators  $A_1$  and  $A_2$ .  $\square$

**Theorem 54.** *Let the operator  $E$  be invertible, and the square pencil admits factorization (33). Then the impulse response  $\mathcal{T}$  and its derivative  $\dot{\mathcal{T}}$  can be represented in the form*

$$\begin{aligned}\mathcal{T}(t) &= (\varphi_1 \boxdot \varphi_2)(\exp_t)C = (\varphi_1 \boxtimes \varphi_2)(\exp_t^{[1]})C, \\ \dot{\mathcal{T}}(t) &= (\varphi_1 \boxdot \varphi_2)(\exp_t^{(1)})C = (\varphi_1 \boxtimes \varphi_2)(\exp_t^{(1)[1]})C,\end{aligned}$$

where  $\exp_t^{(1)[1]}(\lambda, \mu) = \frac{\lambda e^{\lambda t} - \mu e^{\mu t}}{\lambda - \mu}$  for  $\lambda \neq \mu$ .

*Proof.* The proof follows from Proposition 52 and Theorem 45.  $\square$

**Corollary 55.** *The spectra of the transformers  $C \mapsto \mathcal{T}(t)$  and  $C \mapsto \dot{\mathcal{T}}(t)$  in the algebra  $\mathbf{B}(\mathbf{B}(Y, X))$  are equal to*

$$\left\{ \exp_t^{[1]}(\lambda, \mu) : \lambda \in \sigma(A), \mu \in \sigma(B) \right\}, \quad \left\{ \exp_t^{(1)[1]}(\lambda, \mu) : \lambda \in \sigma(A), \mu \in \sigma(B) \right\},$$

respectively.

*Proof.* The proof follows from Theorems 39 and 54.  $\square$

*Remark 1.* (a) In article [78], for the approximate calculation of expressions of the type  $(\varphi_1 \boxtimes \varphi_2)(\exp_t^{[1]})$  it is suggested to use the following representation (written in other notations and verified directly)

$$\exp_t^{[1]}(\lambda, \mu) = (e^{\lambda t} + e^{\mu t}) \frac{\tanh\left(\frac{\lambda - \mu}{2}t\right)}{\frac{\lambda - \mu}{2}}, \quad \lambda \neq \mu.$$

By Theorem 32,  $(\varphi_1 \boxtimes \varphi_2)(e^{\lambda t} + e^{\mu t})$  is  $\varphi_1(\exp_t) \boxtimes \mathbf{1} + \mathbf{1} \boxtimes \varphi_2(\exp_t)$ . By Corollary 37, the operator  $(\varphi_1 \boxtimes \varphi_2) \left( \frac{\tanh\left(\frac{\lambda - \mu}{2}t\right)}{\frac{\lambda - \mu}{2}} \right)$  is the function  $\tau(z) = \frac{\tanh\left(\frac{z}{2}\right)}{\frac{z}{2}}$  of the transformer  $(A \boxtimes \mathbf{1} - \mathbf{1} \boxtimes B)t$ . The function  $\tau$  is analytic in the circle  $|z| < \pi$ . In [78], for its computation, it is suggested to use the Taylor polynomials or rational approximations.

(b) Formulae

$$\exp_t^{[1]}(\lambda, \mu) = e^{\frac{(\lambda + \mu)t}{2}} \frac{\sinh\left(\frac{\lambda - \mu}{2}t\right)}{\frac{\lambda - \mu}{2}}, \quad \exp_t^{[1]}(\lambda, \mu) = e^{\mu t} \frac{e^{(\lambda - \mu)t} - 1}{\lambda - \mu},$$

assuming a similar usage, are suggested in book [65, formulae (10.17)].

(c) We present a formula that enables one to apply similar ideas for the calculation of  $\exp_t^{(1)[1]}$ :

$$\begin{aligned}\exp_t^{(1)[1]}(\lambda, \mu) &= \frac{\lambda e^{\lambda t} - \mu e^{\mu t}}{\lambda - \mu} = \frac{\lambda e^{\lambda t} - \mu e^{\lambda t} + \mu e^{\lambda t} - \mu e^{\mu t}}{\lambda - \mu} = e^{\lambda t} + \mu \frac{e^{\lambda t} - e^{\mu t}}{\lambda - \mu} = \\ &= e^{\lambda t} + \mu \exp_t^{[1]}(\lambda, \mu).\end{aligned}\tag{34}$$

**Corollary 56.** *Let  $E = \mathbf{1}$ . Then*

$$\mathcal{T}(t+s) = \mathcal{T}_1(t)\mathcal{T}(s) + \mathcal{T}(t)\mathcal{T}_2(s),$$

where

$$\begin{aligned}\mathcal{T}_1(t) &= \varphi_1(\exp_t) = \frac{1}{2\pi i} \int_{\Gamma_1} \exp_t(\lambda) R_{1,\lambda} d\lambda, \\ \mathcal{T}_2(t) &= \varphi_2(\exp_t) = \frac{1}{2\pi i} \int_{\Gamma_2} \exp_t(\mu) R_{2,\mu} d\mu.\end{aligned}$$

*Proof.* This is a special case of Corollary 49.  $\square$

Issues related to factorization are widely discussed in the literature [27, 76, 81, 83, 90, 96, 117, 125]. The factorization of an operator pencils of an arbitrary order is discussed in [50, 50, 56, 58, 72, 87, 96, 97, 128, 129].

Estimates of the norms of operators  $(\varphi_1 \boxtimes \varphi_2)(f)C$  are obtained in [44, 45]; special attention is paid to  $\mathcal{T}(t)$  and  $\dot{\mathcal{T}}(t)$ . Estimates of the norm of  $e^{(A \otimes \mathbf{1} + \mathbf{1} \otimes B)t}$  are given in [12].

## 10. THE TRANSFORMATOR $Q$ AND THE SYLVESTER EQUATION

It often arises the problem of calculating the transformator  $Q = (\varphi_1 \boxtimes \varphi_2)w$ , where

$$w(\lambda, \mu) = \frac{1}{\lambda - \mu}.$$

As a rule, it is equivalent to solving the Sylvester equation. In this Section, we discuss some properties of the transformator  $Q$ .

Let  $X \boxtimes Y$  be an extended tensor product of Banach spaces  $X$  and  $Y$ , and  $R_{1,(\cdot)}$  and  $R_{2,(\cdot)}$  be pseudo-resolvents in the algebras  $\mathbf{B}_0(X)$  and  $\mathbf{B}_0(Y)$  respectively. We assume that the extended singular sets  $\bar{\sigma}(R_{1,(\cdot)})$  and  $\bar{\sigma}(R_{2,(\cdot)})$  are disjoint. We consider the transformator  $Q$  defined as

$$Q = (\varphi_1 \boxtimes \varphi_2)w,$$

where<sup>8</sup>

$$w(\lambda, \mu) = \frac{1}{\lambda - \mu}.$$

If necessary, we will use the more detailed notation  $Q_{\varphi_1, \varphi_2}$  or  $Q_{A, B}$ .

**Proposition 57.** *We assume that the extended singular sets  $\bar{\sigma}(R_{1,(\cdot)})$  and  $\bar{\sigma}(R_{2,(\cdot)})$  of the pseudo-resolvents  $R_{1,(\cdot)}$  and  $R_{2,(\cdot)}$  are disjoint. Then*

$$Q = \frac{1}{2}(\varphi_1 \boxtimes \varphi_2)(\operatorname{sgn}_{1|2}),\tag{35}$$

---

<sup>8</sup>The function  $w$  is meromorphic with the point of indeterminacy  $(\infty, \infty)$ .



where the function  $\text{sgn}_{1|2}$  is equal to 1 in a neighborhood of the extended singular set  $\bar{\sigma}(R_{1,(\cdot)})$  and is equal to  $-1$  in a neighborhood of the extended singular set  $\bar{\sigma}(R_{2,(\cdot)})$ . The transformator  $Q$  can be represented in the form

$$Q = \frac{1}{2\pi i} \int_{\Gamma} R_{1,\lambda} \boxtimes R_{2,\lambda} d\lambda, \quad (36)$$

where  $\Gamma$  is an oriented envelope of  $\bar{\sigma}(R_{1,(\cdot)})$  with respect to  $\bar{\sigma}(R_{2,(\cdot)})$ .



FIGURE 3. Various options of an arrangement of the contours  $\Gamma_1$  and  $\Gamma_2$  and the extended singular sets

*Proof.* It is easy to verify that  $\text{sgn}_{1|2}^{[1]} = 2w$ . So, formula (35) follows from Theorem 45.

We calculate  $(\varphi_1 \boxtimes \varphi_2)w$ . To be definite, we assume that  $\infty \notin \bar{\sigma}(R_{2,(\cdot)})$ . We assume that the oriented envelope  $\Gamma_1$  of the set  $\bar{\sigma}(R_{1,(\cdot)})$  and the oriented envelope  $\Gamma_2$  of the set  $\sigma(R_{2,(\cdot)})$  are located as shown in Fig. 3. In particular,  $\lambda - \mu$  is not equal to zero for  $\lambda \in \Gamma_1$  and  $\mu \in \Gamma_2$ . We have (note that in representation (20) for the function  $w$ , we have  $\delta_1 = \delta_2 = 0$ )

$$\begin{aligned} Q &= (\varphi_1 \boxtimes \varphi_2)w = \frac{1}{(2\pi i)^2} \int_{\Gamma_1} \int_{\Gamma_2} \frac{1}{\lambda - \mu} R_{1,\lambda} \boxtimes R_{2,\mu} d\mu d\lambda \\ &= \frac{1}{2\pi i} \int_{\Gamma_1} R_{1,\lambda} \boxtimes \left( \frac{1}{2\pi i} \int_{\Gamma_2} \frac{R_{2,\mu}}{\lambda - \mu} d\mu \right) d\lambda = \frac{1}{2\pi i} \int_{\Gamma_1} R_{1,\lambda} \boxtimes R_{2,\lambda} d\lambda. \end{aligned}$$

Obviously,  $\Gamma_1$  is an oriented envelope of  $\bar{\sigma}(R_{1,(\cdot)})$  with respect to  $\sigma(R_{2,(\cdot)})$ .  $\square$

**Proposition 58** ([13, 63], [86, Lemma 2.2]). *Let  $A \in \mathbf{B}(X)$ ,  $B \in \mathbf{B}(Y)$ , and the embeddings  $\sigma(A) \subset \{\lambda \in \mathbb{C} : \text{Re } \lambda < \rho\}$  and  $\sigma(B) \subset \{\lambda \in \mathbb{C} : \text{Re } \lambda > \rho\}$  hold for some  $\rho \in \mathbb{R}$ . Then*

$$Q = - \int_0^\infty e^{At} \boxtimes e^{-Bt} dt.$$

*Proof.* We begin with the identity

$$w(\lambda, \mu) = - \int_0^\infty e^{\lambda t} e^{-\mu t} dt.$$

It is valid for  $\lambda \in U$  and  $\mu \in V$  provided the neighborhoods  $U \supset \sigma(A)$  and  $V \supset \sigma(B)$  are sufficiently small. Moreover, we may assume that the integral converges uniformly for  $\lambda \in U$  and  $\mu \in V$ . We substitute this integral into the formula

$$(\varphi_1 \boxtimes \varphi_2)w = \frac{1}{(2\pi i)^2} \int_{\Gamma_1} \int_{\Gamma_2} w(\lambda, \mu) R_{A,\lambda} \boxtimes R_{B,\mu} d\mu d\lambda$$

from Theorem 32 assuming that  $\Gamma_1 \subset U$  and  $\Gamma_2 \subset V$ . By the uniform convergence of the last integral, we may change the order of integration:

$$\begin{aligned} (\varphi_1 \boxtimes \varphi_2)(w) &= \frac{1}{(2\pi i)^2} \int_{\Gamma_1} \int_{\Gamma_2} \left( - \int_0^\infty e^{\lambda t} e^{-\mu t} dt \right) R_{A,\lambda} \boxtimes R_{B,\mu} d\mu d\lambda \\ &= - \int_0^\infty \left( \frac{1}{(2\pi i)^2} \int_{\Gamma_1} \int_{\Gamma_2} e^{\lambda t} e^{-\mu t} R_{A,\lambda} \boxtimes R_{B,\mu} d\mu d\lambda \right) dt \\ &= - \int_0^\infty e^{At} \boxtimes e^{-Bt} dt. \quad \square \end{aligned}$$

**Proposition 59** ([13, Theorem 9.1]). *Let the embeddings  $\bar{\sigma}(R_{1,(\cdot)}) \subset \{\lambda \in \mathbb{C} : |\lambda| < \rho\}$  and  $\bar{\sigma}(R_{2,(\cdot)}) \subset \{\lambda \in \mathbb{C} : |\lambda| > \rho\}$  hold for some  $\rho > 0$ . Then*

$$Q = - \sum_{n=0}^{\infty} A^n \boxtimes R_{2,0}^{n+1},$$

where the operator  $A \in \mathbf{B}(X)$  generates  $R_{1,(\cdot)}$  according to Proposition 24.

*Proof.* We consider the identity

$$w(\lambda, \mu) = - \sum_{n=0}^{\infty} \frac{\lambda^n}{\mu^{n+1}}.$$

It is valid for  $\lambda \in U$  and  $\mu \in V$ , where the neighborhoods  $U \supset \sigma(A)$  and  $V \supset \bar{\sigma}(R_{2,(\cdot)})$  are sufficiently small. Moreover, we may assume that the series converges uniformly for  $\lambda \in U$  and  $\mu \in V$ . We substitute this series into the formula

$$(\varphi_1 \boxtimes \varphi_2)(w) = \frac{1}{(2\pi i)^2} \int_{\Gamma_1} \int_{\Gamma_2} w(\lambda, \mu) R_{1,\lambda} \boxtimes R_{2,\mu} d\mu d\lambda + \frac{\delta}{2\pi i} \int_{\Gamma_1} w(\lambda, \infty) R_{1,\lambda} \boxtimes \mathbf{1}_Y d\lambda$$

from Theorem 33 assuming that  $\Gamma_1 \subset U$  and  $\Gamma_2 \subset V$ . By the uniform convergence of the series (and by  $w(\lambda, \infty) = 0$ ), we have

$$\begin{aligned} (\varphi_1 \boxtimes \varphi_2)(w) &= \frac{1}{(2\pi i)^2} \int_{\Gamma_1} \int_{\Gamma_2} - \sum_{n=0}^{\infty} \frac{\lambda^n}{\mu^{n+1}} R_{1,\lambda} \boxtimes R_{2,\mu} d\mu d\lambda \\ &= - \sum_{n=0}^{\infty} \frac{1}{(2\pi i)^2} \int_{\Gamma_1} \int_{\Gamma_2} \frac{\lambda^n}{\mu^{n+1}} R_{1,\lambda} \boxtimes R_{2,\mu} d\mu d\lambda = - \sum_{n=0}^{\infty} A^n \boxtimes R_{2,0}^{n+1}. \quad \square \end{aligned}$$

Theorem 60 below reduces the calculation of  $[(\varphi_1 \boxtimes \varphi_2)f]C$  to the calculation of  $\varphi_1(f)$  and  $\varphi_2(f)$  provided  $Q(C)$  is known; it is a version of Theorem 46.

**Theorem 60.** *Let the extended singular sets  $\bar{\sigma}(R_{1,(\cdot)})$  and  $\bar{\sigma}(R_{2,(\cdot)})$  of the pseudo-resolvents  $R_{1,(\cdot)}$  and  $R_{2,(\cdot)}$  be disjoint, and  $f \in \mathbf{O}(\bar{\sigma}(R_{1,(\cdot)}) \cup \bar{\sigma}(R_{2,(\cdot)}))$ . Then*

$$[(\varphi_1 \boxtimes \varphi_2)f]C = [\varphi_1(f) \boxtimes \mathbf{1} - \mathbf{1} \boxtimes \varphi_2(f)]Q(C).$$

In the special case, where  $\mathbf{B}(Y, X)$  is taken as extended tensor product (see example 3(e)),

$$[(\varphi_1 \boxtimes \varphi_2)f]C = \varphi_1(f) \cdot Q(C) - Q(C) \cdot \varphi_2(f).$$

*Proof.* By Theorem 45,

$$[(\varphi_1 \boxtimes \varphi_2)f]C = [(\varphi_1 \boxtimes \varphi_2)f^{[1]}]C = (\varphi_1 \boxtimes \varphi_2)((f \otimes u)w)C - (\varphi_1 \boxtimes \varphi_2)((u \otimes f)w)C,$$

where  $(f \otimes u)(\lambda, \mu) = f(\lambda)$ ,  $(u \otimes f)(\lambda, \mu) = f(\mu)$ . From Theorems 32, 33, and 34 it follows that

$$\begin{aligned} [(\varphi_1 \boxminus \varphi_2)f]C &= (\varphi_1(f) \boxtimes \mathbf{1})[(\varphi_1 \boxtimes \varphi_2)(w)C] - (\mathbf{1} \boxtimes \varphi_2(f))[(\varphi_1 \boxtimes \varphi_2)(w)C] \\ &= (\varphi_1(f) \boxtimes \mathbf{1})Q(C) - (\mathbf{1} \boxtimes \varphi_2(f))Q(C). \quad \square \end{aligned}$$

In Corollary 61 below, we present a version of Theorem 60. It suggests the reverse order of operations, which enables one to apply the transformator  $Q$  only once; namely, at first,  $\varphi_1(f)$  and  $\varphi_2(f)$  are calculated, and then  $Q(\cdot)$  is applied.

**Corollary 61.** *Let the extended singular sets  $\bar{\sigma}(R_{1,(\cdot)})$  and  $\bar{\sigma}(R_{2,(\cdot)})$  be disjoint. Then*

$$[(\varphi_1 \boxminus \varphi_2)f]C = Q(\varphi_1(f) \cdot C - C \cdot \varphi_2(f)).$$

*Proof.* It is sufficient to note that the transformators  $\varphi_1(f) \boxtimes \mathbf{1}$ ,  $\mathbf{1} \boxtimes \varphi_2(f)$ , and  $Q = (\varphi_1 \boxtimes \varphi_2)(w)$  commute, and then apply Theorem 60.  $\square$

Let  $A \in \mathbf{B}(X)$  and  $B \in \mathbf{B}(Y)$ . The equation

$$AZ - ZB = C \tag{37}$$

for the unknown  $Z \in \mathbf{B}(Y, X)$  with the free term  $C \in \mathbf{B}(Y, X)$  is called the (*continuous*) *Sylvester equation* [13, 70, 119]. The Sylvester equation is connected with a number of applications [5, 10, 23, 42, 71, 102, 120] and is widely discussed in the literature.

**Theorem 62.** *Let  $A \in \mathbf{B}(X)$  and  $B \in \mathbf{B}(Y)$ . The Sylvester equation (37) has a unique solution  $Z \in \mathbf{B}(Y, X)$  for any  $C \in \mathbf{B}(Y, X)$  if and only if the spectra of the operators  $A$  and  $B$  are disjoint. This solution coincides with the operator  $Q(C)$ .*

*Proof.* By Theorem 32 and Corollary 35, the transformator  $Z \mapsto AZ - ZB$  is equal to  $(\varphi_1 \boxtimes \varphi_2)f$ , where  $f(\lambda, \mu) = \lambda - \mu$ . By Theorem 39, its spectrum is equal to  $\sigma(A) - \sigma(B)$ . Therefore the transformator is invertible if and only if  $0 \notin \sigma(A) - \sigma(B)$ . By Theorem 32, the inverse transformator is  $(\varphi_1 \boxtimes \varphi_2)w$ , where  $w(\lambda, \mu) = \frac{1}{\lambda - \mu}$ .  $\square$

The equation

$$Z - AZB = C \tag{38}$$

is called the (*discrete*) *Sylvester equation* [70] or the *Stein equation* [85]. Its theory is similar to the theory of equation (37).

**Theorem 63.** *Let  $A \in \mathbf{B}(X)$  and  $B \in \mathbf{B}(Y)$ . The Sylvester equation (38) has a unique solution  $Z \in \mathbf{B}(Y, X)$  for any  $C \in \mathbf{B}(Y, X)$  if and only if the product of the spectra of the operators  $A$  and  $B$  does not contain 1. This solution coincides with the operator  $[(\varphi_1 \boxtimes \varphi_2)(s)](C)$ , where*

$$s(\lambda, \mu) = \frac{1}{1 - \lambda\mu}.$$

*Proof.* By Theorem 32 and Corollary 35, the transformator  $Z \mapsto Z - AZB$  is equal to  $(\varphi_1 \boxtimes \varphi_2)f$ , where  $f(\lambda, \mu) = 1 - \lambda\mu$ . By Theorem 39, its spectrum is equal to  $1 - \sigma(A)\sigma(B)$ . Therefore the transformator is invertible if and only if  $1 \notin \sigma(A)\sigma(B)$ . By Theorem 32, the inverse transformator is  $(\varphi_1 \boxtimes \varphi_2)s$ .  $\square$

*Remark 2.* Let us return to equation (37) and discuss the case where  $A$  and  $B$  are unbounded operators or linear relations. The natural hypothesis is as follows: if the extended singular sets of the resolvents of  $A$  and  $B$  are disjoint (and thus  $A$  or  $B$  is a bounded operator), then equation (37) has a unique solution, which is determined by the transformator  $Q$ . The problem is: How one can interpret equation (37)? We discuss some variants.

First, we assume that  $A$  and  $B$  are linear relations with non-empty resolvent sets, and the extended spectra of  $A$  and  $B$  are disjoint. We assume that  $C \in \mathbf{B}(Y, X)$  and a solution  $Z \in \mathbf{B}(Y, X)$  of equation (37) is of interest.

To begin with, we show that without loss of generality one can assume that the inverse operators of  $A$  and  $B$  are everywhere defined bounded operators. Since the extended spectra of  $A$  and  $B$  are closed and disjoint, there exists  $\nu \notin \bar{\sigma}(A) \cup \bar{\sigma}(B)$ . We rewrite (37) in the form

$$-\nu Z + AZ + \nu Z - ZB = C,$$

and then in the form (with the invertible coefficients  $\nu \mathbf{1} - A$  and  $\nu \mathbf{1} - B$ )

$$-(\nu \mathbf{1} - A)Z + Z(\nu \mathbf{1} - B) = C.$$

See [57, Proposition A.1.1, p. 281] or [92, Theorem 36] for a justification of the last equality in the case of linear relations.

We consider the case where  $\infty \in \bar{\sigma}(A)$ . Since the relation  $A$  is invertible, its range coincides with the whole of  $X$ , and the image of the zero is zero (otherwise the left side of equation (37) is not an operator). So,  $A$  is an operator (not a relation). We call an operator  $Z \in \mathbf{B}(Y, X)$ , whose range is contained in the domain of the operator  $A$  (otherwise the domains of the left and the right sides of equation (37) are different), a *solution* of equation (37) provided it satisfies the equation. Since  $\infty \notin \bar{\sigma}(B)$ ,  $B$  is a bounded linear operator, see Proposition 24. Multiplying (37) by  $A^{-1}$  we obtain

$$Z - A^{-1}ZB = A^{-1}C. \quad (39)$$

By [57, Proposition A.3.1],  $\sigma(A^{-1}) = \{ \frac{1}{\lambda} : \lambda \in \bar{\sigma}(A) \}$ . By Theorem 63, equation (39) has a unique solution  $Z$  for an arbitrary  $A^{-1}C$  if  $0 \notin \{ 1 - \lambda\mu : \lambda \in \sigma(A^{-1}), \mu \in \sigma(B) \} = \{ 1 - \frac{\mu}{\lambda} : \lambda \in \bar{\sigma}(A), \mu \in \sigma(B) \}$ , which is the case, because the extended spectra of  $A$  and  $B$  are disjoint. We multiply (39) on the left by  $A$  (taking into account that  $AA^{-1} = \mathbf{1}$ ). As a result we arrive at the original equation (37). Therefore  $Z$  is a solution of equation (37) as well.

We discuss the case where  $\infty \in \bar{\sigma}(B)$ . We assume that the domain of the relation  $B$  coincides with the whole of  $Y$  (otherwise the domains of the left and right sides of equation (37) are different). Since the relation  $B$  is invertible, its range coincides with the whole of  $Y$ , the kernel is zero, but the image of the zero  $\text{Im}_0 B = \{ x : (0, x) \in B \}$  may consist not only of zero. We call an operator  $Z \in \mathbf{B}(Y, X)$ , whose kernel contains  $\text{Im}_0 B$  (otherwise the left side of equation (37) is not an operator), a *solution* of equation (37) provided it satisfies the equation. Multiplying (37) on the right by  $B^{-1}$  we obtain

$$AZB^{-1} - ZBB^{-1} = CB^{-1}.$$

According to [92, Theorem 16] we rewrite this equation in the form

$$AZB^{-1} - Z\mathbf{1}_{Y:\text{Im}_0 B} = CB^{-1},$$

where  $\mathbf{1}_{Y:\text{Im}_0 B} = \{ (y_1, y_2) \in Y \times Y : (0, y_1 - y_2) \in B \}$ . Since the kernel of the operator  $Z$  contains  $\text{Im}_0 B$ , the last equation can be rewritten as

$$AZB^{-1} - Z = CB^{-1}.$$

By Theorem 63, this equation has a unique solution  $Z$  for an arbitrary  $CB^{-1}$  if  $0 \notin \{ 1 - \lambda\mu : \lambda \in \sigma(A), \mu \in \sigma(B^{-1}) \} = \{ 1 - \frac{\lambda}{\mu} : \lambda \in \sigma(A), \mu \in \bar{\sigma}(B) \}$ , which is the case, because the extended spectra of  $A$  and  $B$  are disjoint. Multiplying the last equation on the right by  $B$  we obtain

$$AZB^{-1}B - ZB = CB^{-1}B,$$

or (according to [92, Theorem 16] and  $\text{Ker } B = 0$ )

$$AZ - ZB = C.$$

So,  $Z$  is a solution of original equation (37) as well.

We consider another case: let  $B$  be an invertible unbounded operator with the dense domain  $\text{Dom } B$  (in particular,  $\infty \in \bar{\sigma}(B)$ ). We call an operator  $Z \in \mathbf{B}(Y, X)$  a *solution* if

$$AZy - ZBy = Cy$$

for all  $y \in \text{Dom } B$ . Let us look for a solution of equation (37) in the form  $Z = VB^{-1}$ , where  $V \in \mathbf{B}(Y, X)$  is a new unknown operator. Substituting  $Z = VB^{-1}$  into equation (37) we obtain

$$AVB^{-1} - VB^{-1}B = C, \tag{40}$$

or

$$AVB^{-1} - V\mathbf{1}_{\text{Dom } B} = C,$$

where  $\mathbf{1}_{\text{Dom } B}$  is the identity operator with the domain  $\text{Dom } B$ . By our definition of a solution, the last equation is equivalent to the equation

$$AVB^{-1} - V = C.$$

Obviously, it has a unique solution  $V$ . Returning to equivalent equation (40) we see that the operator  $Z = VB^{-1}$  is a solution of the original equation.

Theorem 62 for matrices was first proved in [122]. An independent proof of its sufficient part for the case of operators was obtained in [21, 82, 109]. For a Hilbert space, a necessary and sufficient condition for the solvability of the Sylvester equation was first obtained in [26], see also [49, c. 54]. An analogue of Theorem 63 for matrices is proved, for example, in [85].

The representation for the solution of the Sylvester equation in the form of contour integral (36) was first published in [109], see also Example 4. Estimates of the solution of the Sylvester equation are given in [44, 46, 47].

The Sylvester equation (37) with unbounded operator coefficients  $A$  and  $B$  is considered in [2, 3, 39, 79, 94, 104, 116].

A generalization of the transformer  $Q$  ( $Q$  corresponds to the function  $w(\lambda, \mu) = \frac{1}{\lambda - \mu}$ ) is the inverse of the transformer  $v_{n+1}^{[1]}(A, B)$ , where  $v_n^{[1]}(\lambda, \mu) = \lambda^{n-1} + \lambda^{n-2}\mu + \dots + \mu^{n-1}$ . It is discussed in [15, 41, 46].

## 11. THE DIFFERENTIAL OF THE FUNCTIONAL CALCULUS

Let  $X$  be a Banach space. The (*Fréchet*) *differential* of a nonlinear transformer  $f: D(f) \subseteq \mathbf{B}(X) \rightarrow \mathbf{B}(X)$  at a point  $A \in \mathbf{B}(X)$  is defined to be a linear transformer  $df(\cdot, A): \mathbf{B}(X) \rightarrow \mathbf{B}(X)$  depending on the parameter  $A$  that possesses the property

$$f(A + \Delta A) = f(A) + df(\Delta A, A) + o(\|\Delta A\|). \tag{41}$$

We assume that a neighborhood of  $A$  is contained in the domain  $D(f)$  of the transformer  $f$ . We recall standard properties of the differential.

**Proposition 64** ([4, § 2.2.2], [32, 8.2.1]). *Let a transformer  $g: \mathbf{B}(X) \rightarrow \mathbf{B}(X)$  be differentiable at a point  $A \in \mathbf{B}(X)$  and a transformer  $f: \mathbf{B}(X) \rightarrow \mathbf{B}(X)$  be differentiable at the point  $g(A) \in \mathbf{B}(X)$ . Then the composition  $f \circ g$  is differentiable at the point  $A$ , and*

$$d(f \circ g)(\cdot, A) = df[dg(\cdot, A), g(A)].$$

**Corollary 65** ([4, § 2.3.4], [32, 8.2.3]). *Let a transformator  $f: \mathbf{B}(X) \rightarrow \mathbf{B}(X)$  be continuously differentiable (i. e.  $df(\cdot, A)$  depends on  $A$  continuously in norm) in a neighborhood of a point  $A \in \mathbf{B}(X)$  and let the transformator  $df(\cdot, A)$  be invertible. Then the inverse transformator of  $f$  is defined and differentiable in a neighborhood of the point  $B = f(A)$ , and the differential of the inverse transformator is equal to the inverse of the original differential:*

$$df^{-1}(\cdot, B) = [df(\cdot, A)]^{-1}.$$

Let  $A \in \mathbf{B}(X)$  and  $f \in \mathbf{O}(\sigma(A))$ . For the transformator

$$A \mapsto f(A) = \frac{1}{2\pi i} \int_{\Gamma} f(\lambda)(\lambda \mathbf{1} - A)^{-1} d\lambda \quad (42)$$

definition (41) of a differential looks as follows:

$$\frac{1}{2\pi i} \int_{\Gamma} f(\lambda)(\lambda \mathbf{1} - (A + \Delta A))^{-1} d\lambda = \frac{1}{2\pi i} \int_{\Gamma} f(\lambda)(\lambda \mathbf{1} - A)^{-1} d\lambda + df(\Delta A, A) + o(\|\Delta A\|).$$

We note that

$$(\lambda \mathbf{1} - (A + \Delta A))^{-1} = ((\lambda \mathbf{1} - A) - \Delta A)^{-1} = R_{\lambda}(\mathbf{1} - \Delta A \cdot R_{\lambda})^{-1} = (\mathbf{1} - R_{\lambda} \cdot \Delta A)^{-1} R_{\lambda}.$$

Based on this formula, we adopt the following definition.

Let  $R_{(\cdot)}$  be a pseudo-resolvent in the algebra  $\mathbf{B}(X)$ . We call *the perturbation of  $R_{(\cdot)}$  by an operator  $\Delta A \in \mathbf{B}(X)$*  the function

$$T_{\lambda} = R_{\lambda}(\mathbf{1} - \Delta A \cdot R_{\lambda})^{-1} = (\mathbf{1} - R_{\lambda} \cdot \Delta A)^{-1} R_{\lambda}. \quad (43)$$

*Remark 3.* We note shortly an additional reasoning in favor of definition (43). Let  $R_{(\cdot)}$  be the resolvent of a linear relation  $A$ , i. e.  $R_{\lambda} = (\lambda \mathbf{1} - A)^{-1}$ . We show that

$$(\mathbf{1} - R_{\lambda} \cdot \Delta A)^{-1} R_{\lambda} = (\lambda \mathbf{1} - A - \Delta A)^{-1}.$$

Obviously (for details, see [57, Proposition A.1.1] or [92, Proposition 12]),  $(\mathbf{1} - R_{\lambda} \cdot \Delta A)^{-1} R_{\lambda} = [R_{\lambda}^{-1}(\mathbf{1} - R_{\lambda} \cdot \Delta A)]^{-1} = [(\lambda \mathbf{1} - A)(\mathbf{1} - R_{\lambda} \cdot \Delta A)]^{-1}$ . Further, since the image of the operator  $R_{\lambda} \cdot \Delta A$  is contained in the image of  $R_{\lambda}$ , which is equal to the domain of the relation  $A$ , by virtue of [92, Theorem 36(a)], we can develop the internal parentheses:  $[\lambda \mathbf{1} - A - (\lambda \mathbf{1} - A)R_{\lambda} \cdot \Delta A]^{-1} = [\lambda \mathbf{1} - A - (\lambda \mathbf{1} - A)(\lambda \mathbf{1} - A)^{-1} \cdot \Delta A]^{-1}$ . We note that  $(\lambda \mathbf{1} - A)(\lambda \mathbf{1} - A)^{-1}$  is equal to the relation  $\mathbf{1}_{X:\text{Im}_0 A} = \{(x_1, x_2) \in X \times X : (0, x_1 - x_2) \in A\}$ . Obviously,  $(\lambda \mathbf{1} - A - \mathbf{1}_{X:\text{Im}_0 A} \cdot \Delta A)^{-1} = (\lambda \mathbf{1} - A - \Delta A)^{-1}$ .

**Proposition 66.** *For any perturbation  $\Delta A \in \mathbf{B}(X)$  the function*

$$T_{\lambda} = R_{\lambda}(\mathbf{1} - \Delta A \cdot R_{\lambda})^{-1} = (\mathbf{1} - R_{\lambda} \cdot \Delta A)^{-1} R_{\lambda}$$

*is a pseudo-resolvent.*

*Proof.* We verify the Hilbert identity for all  $\lambda$  and  $\mu$  such that  $T_{\lambda}$  and  $T_{\mu}$  are defined. We have

$$\begin{aligned} T_{\lambda} - T_{\mu} + (\lambda - \mu)T_{\lambda}T_{\mu} &= (\mathbf{1} - R_{\lambda} \cdot \Delta A)^{-1} R_{\lambda} - R_{\mu}(\mathbf{1} - \Delta A \cdot R_{\mu})^{-1} \\ &\quad + (\lambda - \mu)(\mathbf{1} - R_{\lambda} \cdot \Delta A)^{-1} R_{\lambda} R_{\mu}(\mathbf{1} - \Delta A \cdot R_{\mu})^{-1} \\ &= (\mathbf{1} - R_{\lambda} \cdot \Delta A)^{-1} [R_{\lambda}(\mathbf{1} - \Delta A \cdot R_{\mu}) - (\mathbf{1} - R_{\lambda} \cdot \Delta A)R_{\mu} \\ &\quad + (\lambda - \mu)R_{\lambda}R_{\mu}] \\ &= (\mathbf{1} - R_{\lambda} \cdot \Delta A)^{-1} [R_{\lambda} - R_{\mu} + (\lambda - \mu)R_{\lambda}R_{\mu}] = 0. \quad \square \end{aligned}$$

We define the *differential*  $df(\cdot, R_{(\cdot)})$  of the mapping (which is a generalization of (42))

$$R_{(\cdot)} \mapsto f(R_{(\cdot)}) = \frac{1}{2\pi i} \int_{\Gamma} f(\lambda) R_{\lambda} d\lambda + \delta f(\infty) \mathbf{1}, \quad (44)$$

by means of the formula

$$\begin{aligned} & \frac{1}{2\pi i} \int_{\Gamma} f(\lambda) R_{\lambda} (\mathbf{1} - \Delta A \cdot R_{\lambda})^{-1} d\lambda + \delta f(\infty) \mathbf{1} \\ &= \frac{1}{2\pi i} \int_{\Gamma} f(\lambda) R_{\lambda} d\lambda + \delta f(\infty) \mathbf{1} + df(\Delta A, R_{(\cdot)}) + o(\|\Delta A\|). \end{aligned}$$

**Theorem 67.** *Let  $R_{(\cdot)}$  be a pseudo-resolvent in the algebra  $\mathbf{B}(X)$ , and  $f \in \mathbf{O}(\bar{\sigma}(R_{(\cdot)}))$ . Then the differential of mapping (44) admits the representation*

$$df(\Delta A, R_{(\cdot)}) = \frac{1}{2\pi i} \int_{\Gamma} f(\lambda) R_{\lambda} \Delta A R_{\lambda} d\lambda.$$

In other words,

$$df(\cdot, R_{(\cdot)}) = (\varphi \square \varphi)(f),$$

where  $\varphi$  is the functional calculus generated by the pseudo-resolvent  $R_{(\cdot)}$ .

*Proof.* We assume that

$$\|\Delta A\| \cdot \|R_{\lambda}\| < 1.$$

By Theorem 1, we have

$$\| [R_{\lambda}(\mathbf{1} - \Delta A \cdot R_{\lambda})^{-1} - R_{\lambda}] - R_{\lambda} \Delta A \cdot R_{\lambda} \| \leq \frac{\|R_{\lambda}\|^3 \cdot \|\Delta A\|^2}{1 - \|R_{\lambda}\| \cdot \|\Delta A\|}.$$

Therefore,

$$\begin{aligned} & \left\| \frac{1}{2\pi i} \int_{\Gamma} f(\lambda) R_{\lambda} (\mathbf{1} - \Delta A \cdot R_{\lambda})^{-1} d\lambda - \frac{1}{2\pi i} \int_{\Gamma} f(\lambda) R_{\lambda} d\lambda - \frac{1}{2\pi i} \int_{\Gamma} f(\lambda) R_{\lambda} \Delta A R_{\lambda} d\lambda \right\| \\ &= \left\| \frac{1}{2\pi i} \int_{\Gamma} f(\lambda) [R_{\lambda}(\mathbf{1} - \Delta A \cdot R_{\lambda})^{-1} - R_{\lambda} - R_{\lambda} \Delta A R_{\lambda}] d\lambda \right\| \\ &\leq \frac{1}{2\pi} \int_{\Gamma} |f(\lambda)| \frac{\|R_{\lambda}\|^3 \cdot \|\Delta A\|^2}{1 - \|R_{\lambda}\| \cdot \|\Delta A\|} |d\lambda| = o(\|\Delta A\|). \quad \square \end{aligned}$$

**Proposition 68** ([14, Theorem 2.1]). *Let  $A \in \mathbf{B}(X)$  and  $f \in \mathbf{O}(\sigma(A))$ . Then*

$$df(A \Delta A - \Delta A A, A) = \varphi_A(f) \Delta A - \Delta A \varphi_A(f),$$

where the functional calculus  $\varphi_A$  is generated by the operator  $A$ .

*Proof.* The proof follows from Theorems 46 and 67. □

**Proposition 69** ([65, Theorem 3.3]). *Let  $g, h \in \mathbf{O}(\bar{\sigma}(R_{(\cdot)}))$ . Then*

$$d(gh)(\Delta A, R_{(\cdot)}) = dg(\Delta A, R_{(\cdot)}) h(R_{(\cdot)}) + g(R_{(\cdot)}) dh(\Delta A, R_{(\cdot)}),$$

where  $(gh)(\lambda) = g(\lambda)h(\lambda)$ .

*Proof.* The proof follows from Corollary 49. □

**Corollary 70** ([110, Theorem 10.36], see also [108]). *Let  $f \in \mathbf{O}(\bar{\sigma}(R_{(\cdot)}))$ . We assume that  $\Delta A$  commutes with at least one value  $R_{\mu}$  of the pseudo-resolvent  $R_{(\cdot)}$ . Then*

$$df(\Delta A, R_{(\cdot)}) = \varphi(f') \Delta A = \Delta A \varphi(f').$$

*Proof.* The proof follows from 51.  $\square$

**Theorem 71.** Let  $R_{(\cdot)}$  be a pseudo-resolvent in the algebra  $\mathbf{B}(X)$  and  $f \in \mathbf{O}(\bar{\sigma}(R_{(\cdot)}))$ . Then the differential of mapping (44) admits the representation

$$df(\Delta A, R_{(\cdot)}) = \frac{1}{(2\pi i)^2} \int_{\Gamma_1} \int_{\Gamma_2} f^{[1]}(\lambda, \mu) R_\lambda \Delta A R_\mu d\mu d\lambda, \quad (45)$$

where the divided difference  $f^{[1]}$  is defined by formula (28), the contours  $\Gamma_1$  and  $\Gamma_2$  are oriented envelopes of the extended singular set  $\bar{\sigma}(R_{(\cdot)})$  with respect to the complement  $\bar{\mathbb{C}} \setminus U$ , and  $U$  is the domain of the function  $f$ .

The spectrum of the transformator  $df(\cdot, R_{(\cdot)}): \mathbf{B}(X) \rightarrow \mathbf{B}(X)$  is given by the formula

$$\sigma[df(\cdot, R_{(\cdot)})] = \{ f^{[1]}(\lambda, \mu) : \lambda, \mu \in \bar{\sigma}(R_{(\cdot)}) \}. \quad (46)$$

*Proof.* Theorem 67 shows that  $df(\Delta A, R_{(\cdot)})$  is the operator  $[(\varphi \boxdot \varphi)(f)]\Delta A$ . By Theorem 45,

$$[(\varphi \boxdot \varphi)(f)]\Delta A = [(\varphi \boxtimes \varphi) f^{[1]}]\Delta A.$$

It remains to apply Theorem 32.

Formula (46) follows from Theorem 39.  $\square$

*Example 6.* Let  $A \in \mathbf{B}(X)$ . The following corollaries are consequences of Example 5 and Theorem 71.

The differential of the transformator  $v_2(A) = A^2$  is given by the formula

$$dv_2(\Delta A, A) = A \cdot \Delta A + \Delta A \cdot A, \quad (47)$$

and its spectrum at a point  $A \in \mathbf{B}(X)$  is equal to

$$\sigma[dv_2(\cdot, A)] = \{ \lambda + \mu : \lambda, \mu \in \sigma(A) \}.$$

The differential of the mapping  $r_1(R_{(\cdot)}) = R_{\lambda_0}$  is given by the formula

$$dr_1(\Delta A, R_{(\cdot)}) = R_{\lambda_0} \cdot \Delta A \cdot R_{\lambda_0},$$

and its spectrum at a point  $R_{(\cdot)}$  is equal to

$$\sigma[dr_1(\cdot, R_{(\cdot)})] = \left\{ \frac{1}{(\lambda_0 - \lambda)(\lambda_0 - \mu)} : \lambda, \mu \in \bar{\sigma}(R_{(\cdot)}) \right\}.$$

The differential of the transformator  $\exp_t(A) = e^{At}$  is given by the formula

$$d\exp_t(\Delta A, A) = \sum_{n=0}^{\infty} \frac{t^n}{(n+1)!} \sum_{i=0}^n A^{n-i} \Delta A A^i,$$

and its spectrum at a point  $A \in \mathbf{B}(X)$  is equal to

$$\sigma[d\exp_t(\cdot, A)] = \{ \exp_t^{[1]}(\lambda, \mu) : \lambda, \mu \in \sigma(A) \}.$$

The differential of the transformator  $\exp_t^{(1)}(A) = A e^{At}$  is given by the formula

$$d\exp_t^{(1)}(\Delta A, A) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{i=0}^n A^{n-i} \Delta A A^i.$$

and its spectrum at a point  $A \in \mathbf{B}(X)$  is equal to

$$\sigma[d\exp_t^{(1)}(\cdot, A)] = \{ \exp_t^{(1)[1]}(\lambda, \mu) : \lambda, \mu \in \sigma(A) \}.$$

We note special formulae for the differentials of the transformers  $\exp_t(A) = e^{At}$  and  $\exp_t^{(1)}(A) = A e^{At}$ .



**Proposition 72.** *The differentials of the transformers  $\exp_t(A) = e^{At}$  and  $\exp_t(A)^{(1)} = Ae^{At}$  at a point  $A \in \mathbf{B}(X)$  can be calculated by means of the formulae*

$$d\exp_t(\Delta A, A) = \int_0^t e^{(t-s)A} \Delta A e^{sA} ds = \int_0^t \exp_{t-s}(A) \Delta A \exp_s(A) ds, \quad (48)$$

$$d\exp_t^{(1)}(\Delta A, A) = \exp_t(A) \Delta A + \int_0^t \exp_{t-s}(A) \Delta A A \exp_s(A) ds. \quad (49)$$

*Proof.* For  $\lambda \neq \mu$  we have

$$\frac{e^{\lambda t} - e^{\mu t}}{\lambda - \mu} = \frac{1}{\lambda - \mu} e^{\lambda s} e^{\mu(t-s)} \Big|_{s=0}^{s=t} = \frac{1}{\lambda - \mu} \int_0^t \frac{d}{ds} [e^{\lambda s} e^{\mu(t-s)}] ds = \int_0^t e^{\lambda s} e^{\mu(t-s)} ds.$$

By the continuity, the same representation of  $\exp_t^{[1]}(\lambda, \mu)$  holds for all finite  $\lambda$  and  $\mu$ . Hence, from Theorems 71, 32, and 39 it follows (48). Formula (49) follows from (34).  $\square$

The differentials of inverse functions are defined by the inverse transformers, see Corollary 65. To calculate them and their spectra, one can use the following theorem.

**Theorem 73.** *Let  $R_{(\cdot)}$  be a resolvent of an operator  $A \in \mathbf{B}(X)$ ,  $f \in \mathbf{O}(\sigma(A))$ , and  $0 \notin \bigcup \{ f^{[1]}(\lambda, \mu) : \lambda, \mu \in \sigma(A) \}$ . Then the differential of the transformer  $B \mapsto f^{-1}(B)$  at the point  $B = f(A)$  is given by the formula*

$$df^{-1}(\Delta B, B) = \frac{1}{(2\pi i)^2} \int_{\Gamma_1} \int_{\Gamma_2} \frac{1}{f^{[1]}(\lambda, \mu)} R_\lambda \Delta B R_\mu d\mu d\lambda, \quad (50)$$

where the contours  $\Gamma_1$  and  $\Gamma_2$  are oriented envelopes of the spectrum  $\sigma(A)$  with respect to the point  $\infty$  and the complement of the domain of the function  $f$ .

The spectrum of the transformer  $df^{-1}(\cdot, B) : \mathbf{B}(X) \rightarrow \mathbf{B}(X)$  is given by the formula

$$\sigma[df^{-1}(\cdot, B)] = \bigcup \left\{ \frac{1}{f^{[1]}(\lambda, \mu)} : \lambda, \mu \in \sigma(A) \right\}. \quad (51)$$

*Proof.* By Theorem 71,

$$df(\Delta A, A) = \frac{1}{(2\pi i)^2} \int_{\Gamma_1} \int_{\Gamma_2} f^{[1]}(\lambda, \mu) R_\lambda \Delta A R_\mu d\mu d\lambda, \\ \sigma[df(\cdot, A)] = \{ f^{[1]}(\lambda, \mu) : \lambda, \mu \in \sigma(A) \}.$$

Since  $0 \notin \bigcup \{ f^{[1]}(\lambda, \mu) : \lambda, \mu \in \sigma(A) \}$ , the transformer  $df(\cdot, A)$  is invertible. By Corollary 65, the inverse transformer is the differential of the mapping  $B \mapsto f^{-1}(B)$ , which is the inverse mapping of  $A \mapsto f(A)$ . By Theorem 32, the inverse transformer is given by formula (50). By Theorem 39, its spectrum is given by formula (51).  $\square$

*Example 7.* Let the spectrum of an operator  $B \in \mathbf{B}(X)$  be contained in  $\mathbb{C} \setminus (-\infty, 0]$ . The following corollaries are consequences of Example 6 and Theorem 73.

From (47) it is clear that the differential  $dv_{1/2}(\Delta B, B)$  of the transformer  $v_{1/2}(B) = \sqrt{B}$  satisfies the Sylvester equation

$$\sqrt{B} \cdot dv_{1/2}(\Delta B, B) + dv_{1/2}(\Delta B, B) \sqrt{B} = \Delta B.$$

Therefore,

$$dv_{1/2}(\Delta B, B) = Q_{\sqrt{B}, -\sqrt{B}}(\Delta B),$$

where the transformator  $Q_{\sqrt{B}, -\sqrt{B}}$  is constructed by means of the functional calculus generated by the operators  $\sqrt{B}$  and  $-\sqrt{B}$ . The spectrum of the differential of the transformator  $v_{1/2}(B) = \sqrt{B}$  at the point  $B$  is equal to

$$\sigma[dv_{1/2}(\cdot, B)] = \left\{ \frac{1}{\lambda + \mu} : \lambda, \mu \in \sqrt{\sigma(B)} \right\}.$$

The spectrum of the differential of the transformator  $f : B \mapsto \ln B$ , where  $\ln$  denotes the principal value, at the point  $B$  is equal to

$$\sigma[df(\cdot, B)] = \left\{ \frac{1}{\exp_1^{[1]}(\lambda, \mu)} : \lambda, \mu \in \ln(\sigma(B)) \right\}.$$

*Remark 4.* The equation

$$AZ + ZB + ZCZ + D = 0 \quad (52)$$

with  $A, B, C, D \in \mathbf{B}(X)$  and unknown  $Z \in \mathbf{B}(X)$  is called [70, 79, 80] the *Riccati equation*. It arises in control theory [5, 103]. The differential  $dZ = dZ(\Delta A, \Delta B, \Delta C, \Delta D; Z)$  of the solution  $Z$  of Riccati equation (52) satisfies [70, p. 135] the continuous Sylvester equation

$$(A + ZC)dZ + dZ(B + CZ) = -\Delta D - \Delta A Z - Z \Delta B - Z \Delta C Z.$$

So,

$$dZ(\Delta A, \Delta B, \Delta C, \Delta D; Z) = Q_{A+ZC, -B-CZ}(-\Delta D - \Delta A Z - Z \Delta B - Z \Delta C Z).$$

For pseudo-resolvents generated by bounded operators, Theorem 67 is proved in [110, Theorem 10.38], see also [14, formula (2.3)], [22] and [121]. Representation (45) for matrices is given in [85, Theorem 5.1]. Formula (46) for matrices is proved in [65, Theorem 3.9], its weaker version was previously obtained in [77, Lemma 2.1]. Formula (48) was first obtained in [74, formula (1.8)], see also [10, ch. 10, § 14], [65, formula (10.15)], [77, example 2], [98], [100], [126]. Differentials connected with some specific functions  $f$  are investigated in [1, 29, 65, 66, 77, 78, 126, 130]; a special attention is paid to estimates of their norms which helps to find the condition number of the transformator  $A \mapsto f(A)$ . Properties of differentials of higher orders are investigated in [14].

## REFERENCES

- [1] Al-Mohy A. H., Higham N. J., Relton S. D. Computing the Fréchet derivative of the matrix logarithm and estimating the condition number // SIAM J. Sci. Comput. — 2013. — Vol. 35, no. 4. — P. 394–410. — URL: <http://dx.doi.org/10.1137/120885991>.
- [2] Alberverio S., Makarov K. A., Motovilov A. K. Graph subspaces and the spectral shift function // Canad. J. Math. — 2003. — Vol. 55, no. 3. — P. 449–503. — URL: <http://dx.doi.org/10.4153/CJM-2003-020-7>.
- [3] Alberverio S., Motovilov A. K. Operator Stieltjes integrals with respect to a spectral measure and solutions of some operator equations // Trans. Moscow Math. Soc. — 2011. — Vol. 72, no. 1. — P. 45–77.
- [4] Alekseev V. M., Tikhomirov V. M., Fomin S. V. Optimal control. Contemporary Soviet Mathematics. — Consultants Bureau, New York, 1987. — xiv+309 p. — ISBN: 0-306-10996-4. — URL: <http://dx.doi.org/10.1007/978-1-4615-7551-1>.
- [5] Antoulas A. C. Approximation of large-scale dynamical systems. — Philadelphia, PA : Society for Industrial and Applied Mathematics (SIAM), 2005. — Vol. 6 of Advances in Design and Control. — xxvi+479 p. — ISBN: 0-89871-529-6. — URL: <http://dx.doi.org/10.1137/1.9780898718713>.

- [6] Arendt W. Approximation of degenerate semigroups // Taiwanese J. Math. — 2001. — Vol. 5, no. 2. — P. 279–295.
- [7] Arens R. The adjoint of a bilinear operation // Proc. Amer. Math. Soc. — 1951. — Vol. 2. — P. 839–848.
- [8] Arens R. Operational calculus of linear relations // Pacific J. Math. — 1961. — Vol. 11. — P. 9–23.
- [9] Baskakov A. G. Representation theory for Banach algebras, Abelian groups, and semigroups in the spectral analysis of linear operators // Journal of Mathematical Sciences. — 2006. — Vol. 137, no. 4. — P. 4885–5036.
- [10] Bellman R. Introduction to matrix analysis. — Second edition. — New York-Düsseldorf-London : McGraw-Hill Book Co., 1970. — xxiii+403 p.
- [11] Benzi M., Simoncini V. Approximation of functions of large matrices with Kronecker structure // arXiv:1503.02615. — 2015. — P. 1–21.
- [12] Benzi M., Simoncini V. Decay bounds for functions of matrices with banded or Kronecker structure // arXiv:1501.07376. — 2015. — P. 1–20.
- [13] Bhatia R., Rosenthal P. How and why to solve the operator equation  $AX - XB = Y$  // Bull. London Math. Soc. — 1997. — Vol. 29, no. 1. — P. 1–21. — URL: <http://dx.doi.org/10.1112/S0024609396001828>.
- [14] Bhatia R., Sinha K. B. Derivations, derivatives and chain rules // Linear Algebra Appl. — 1999. — Vol. 302/303. — P. 231–244.
- [15] Bhatia R., Uchiyama M. The operator equation  $\sum_{i=0}^n A^{n-i}XB^i = Y$  // Expo. Math. — 2009. — Vol. 27, no. 3. — P. 251–255. — URL: <http://dx.doi.org/10.1016/j.exmath.2009.02.001>.
- [16] Bichegkuev M. S. Conditions for solubility of difference inclusions // Izv. Math. — 2008. — Vol. 72, no. 4. — P. 647–658.
- [17] Bourbaki N. Éléments de mathématique. Fasc. X. Première partie. Livre III: Topologie générale. Chapitre 10: Espaces fonctionnels. Deuxième édition, entièrement refondue. Actualités Scientifiques et Industrielles, No. 1084. — Paris : Hermann, 1961. — 96 p.
- [18] Bourbaki N. Éléments de mathématique. Fasc. XXXII. Théories spectrales. Chapitre I: Algèbres normées. Chapitre II: Groupes localement compacts commutatifs. Actualités Scientifiques et Industrielles, No. 1332. — Paris : Hermann, 1967. — iv+166 p.
- [19] Brown A., Pearcy C. Spectra of tensor products of operators // Proc. Amer. Math. Soc. — 1966. — Vol. 17. — P. 162–166.
- [20] Cross R. Multivalued linear operators. — New York : Marcel Dekker, Inc., 1998. — Vol. 213 of Monographs and Textbooks in Pure and Applied Mathematics. — x+335 p. — ISBN: 0-8247-0219-0.
- [21] Daleckiĭ Y. L. On the asymptotic solution of a vector differential equation // Doklady Akad. Nauk SSSR (New Series). — 1953. — Vol. 92. — P. 881–884 (in Russian).
- [22] Daleckiĭ Y. L. Differentiation of non-Hermitian matrix functions depending on a parameter // Amer. Math. Soc. Transl., Series 2. — 1965. — Vol. 47. — P. 73–87.
- [23] Daleckiĭ Y. L., Kreĭn M. G. Stability of solutions of differential equations in Banach space. — Providence, RI : American Mathematical Society, 1974. — Vol. 43 of Translations of Mathematical Monographs. — vi+386 p.
- [24] Davies P. I., Higham N. J. A Schur-Parlett algorithm for computing matrix functions // SIAM J. Matrix Anal. Appl. — 2003. — Vol. 25, no. 2. — P. 464–485 (electronic). — URL: <http://dx.doi.org/10.1137/S0895479802410815>.

- [25] Davis C. Explicit functional calculus // Linear Algebra and Appl. — 1973. — Vol. 6. — P. 193–199.
- [26] Davis C., Rosenthal P. Solving linear operator equations // Canad. J. Math. — 1974. — Vol. 26. — P. 1384–1389.
- [27] Davis G. J. Numerical solution of a quadratic matrix equation // SIAM J. Sci. Statist. Comput. — 1981. — Vol. 2, no. 2. — P. 164–175. — URL: <http://dx.doi.org/10.1137/0902014>.
- [28] Defant A., Floret K. Tensor norms and operator ideals. — Amsterdam : North-Holland Publishing Co., 1993. — Vol. 176 of North-Holland Mathematics Studies. — xii+566 p. — ISBN: 0-444-89091-2.
- [29] Dieci L., Morini B., Papini A. Computational techniques for real logarithms of matrices // SIAM J. Matrix Anal. Appl. — 1996. — Vol. 17, no. 3. — P. 570–593. — URL: <http://dx.doi.org/10.1137/S0895479894273614>.
- [30] Dieci L., Papini A. Padé approximation for the exponential of a block triangular matrix // Linear Algebra Appl. — 2000. — Vol. 308, no. 1-3. — P. 183–202. — URL: [http://dx.doi.org/10.1016/S0024-3795\(00\)00042-2](http://dx.doi.org/10.1016/S0024-3795(00)00042-2).
- [31] Diestel J., Fourie J. H., Swart J. The metric theory of tensor products. — Providence, RI : American Mathematical Society, 2008. — x+278 p. — ISBN: 978-0-8218-4440-3. — Grothendieck's résumé revisited. URL: <http://dx.doi.org/10.1090/mbk/052>.
- [32] Dieudonné J. Foundations of modern analysis. Pure and Applied Mathematics, Vol. X. — New York–London : Academic Press, 1960. — xiv+361 p.
- [33] Dunford N. An ergodic theorem for  $n$ -parameter groups // Proceedings of the National Academy of Sciences. — 1939. — Vol. 25, no. 4. — P. 195–196.
- [34] Dunford N. Spectral theory // Bull. Amer. Math. Soc. — 1943. — Vol. 49. — P. 637–651.
- [35] Dunford N. Spectral theory. I. Convergence to projections // Trans. Amer. Math. Soc. — 1943. — Vol. 54. — P. 185–217.
- [36] Dunford N., Schwartz J. T. Linear operators. Part I. General theory. Wiley Classics Library. — New York : John Wiley & Sons, Inc., 1988. — xiv+858 p.
- [37] Embry M. R., Rosenblum M. Spectra, tensor products, and linear operator equations // Pacific J. Math. — 1974. — Vol. 53. — P. 95–107.
- [38] Favini A., Yagi A. Degenerate differential equations in Banach spaces. — New York–Basel–Hong Kong : CRC Press, 1998. — xii+314 p. — ISBN: 0-8247-1677-9.
- [39] Freeman J. M. The tensor product of semigroups and the operator equation  $SX - XT = A$ . // J. Math. Mech. — 1969/1970. — Vol. 19. — P. 819–828.
- [40] Frommer A., Simoncini V. Matrix functions // Model order reduction: theory, research aspects and applications. — Berlin : Springer, 2008. — Vol. 13 of Math. Ind. — P. 275–303. — URL: [http://dx.doi.org/10.1007/978-3-540-78841-6\\_13](http://dx.doi.org/10.1007/978-3-540-78841-6_13).
- [41] Furuta T. Positive semidefinite solutions of the operator equation  $\sum_{j=1}^n A^{n-j} X A^{j-1} = B$  // Linear Algebra Appl. — 2010. — Vol. 432, no. 4. — P. 949–955. — URL: <http://dx.doi.org/10.1016/j.laa.2009.10.008>.
- [42] Gantmacher F. R. The theory of matrices. Vol. 1. — Providence, RI : AMS Chelsea Publishing, 1998. — x+374 p. — ISBN: 0-8218-1376-5.
- [43] Gel'fond A. O. Calculus of finite differences. International Monographs on Advanced Mathematics and Physics. — Delhi : Hindustan Publishing Corp., 1971. — vi+451 p.
- [44] Gil' M. Norm estimates for functions of two non-commuting matrices // Electron. J. Linear Algebra. — 2011. — Vol. 22. — P. 504–512. — URL: <http://dx.doi.org/10.13001/1081-3810.1453>.

- [45] Gil' M. Norm estimates for functions of two non-commuting operators // Rocky Mountain J. Math. — 2015. — Vol. 45, no. 3. — P. 927–940. — URL: <http://dx.doi.org/10.1216/RMJ-2015-45-3-927>.
- [46] Gil' M. Norm estimates for solutions of polynomial operator equations // J. Math. — 2015. — no. Article ID 524829. — P. 7. — URL: <http://dx.doi.org/10.1155/2015/524829>.
- [47] Gil' M. Resolvents of operators on tensor products of Euclidean spaces // Linear and Multilinear Algebra. — 2015. — P. 1–18.
- [48] Gil' M. I. Regular functions of operators on tensor products of Hilbert spaces // Integral Equations Operator Theory. — 2006. — Vol. 54, no. 3. — P. 317–331. — URL: <http://dx.doi.org/10.1007/s00020-004-1359-8>.
- [49] Gohberg I., Goldberg S., Kaashoek M. A. Classes of linear operators. I. — Basel : Birkhäuser Verlag, 1990. — Vol. 49 of Operator Theory: Advances and Applications. — xiv+468 p. — ISBN: 3-7643-2531-3. — URL: <http://dx.doi.org/10.1007/978-3-0348-7509-7>.
- [50] Gohberg I., Lancaster P., Rodman L. Matrix polynomials. Computer Science and Applied Mathematics. — New York–London : Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], 1982. — xiv+409 p. — ISBN: 0-12-287160-X.
- [51] Gohberg I. C., Kreĭn M. G. Theory and applications of Volterra operators in Hilbert space. — Providence, RI : American Mathematical Society, 1970. — Vol. 24 of Translations of Mathematical Monographs. — x+430 p. — ISBN: 0821815741, 9780821815748. — URL: <http://gen.lib.rus.ec/book/index.php?md5=b3b7a4a44343a7b57a1f0a0aec37ce34>.
- [52] Golub G. H., Van Loan C. F. Matrix computations. Johns Hopkins Studies in the Mathematical Sciences. — Third edition. — Baltimore, MD : Johns Hopkins University Press, 1996. — xxx+698 p. — ISBN: 0-8018-5413-X; 0-8018-5414-8.
- [53] Graham A. Kronecker products and matrix calculus: with applications. Ellis Horwood Series in Mathematics and its Applications. — New York : Ellis Horwood Ltd., Chichester; Halsted Press [John Wiley & Sons, Inc.], 1981. — 130 p. — ISBN: 0-85312-391-8.
- [54] Greene R., Krantz S. Function theory of one complex variable. — Third edition. — Providence, RI : Amer. Math. Soc., 2006. — Vol. 40 of Graduate Studies in Mathematics. — x+504 p. — URL: <http://dx.doi.org/10.1090/gsm/040>.
- [55] Grothendieck A. Produits tensoriels topologiques et espaces nucléaires. Mem. Amer. Math. Soc. no. 16. — Providence, RI : American Mathematical Society, 1966. — 383 p.
- [56] Guo C.-H., Higham N. J., Tisseur F. Detecting and solving hyperbolic quadratic eigenvalue problems // SIAM J. Matrix Anal. Appl. — 2008/09. — Vol. 30, no. 4. — P. 1593–1613. — URL: <http://dx.doi.org/10.1137/070704058>.
- [57] Haase M. The functional calculus for sectorial operators. — Basel : Birkhäuser Verlag, 2006. — Vol. 169 of Operator Theory: Advances and Applications. — xiv+392 p. — ISBN: 978-3-7643-7697-0; 3-7643-7697-X. — URL: <http://dx.doi.org/10.1007/3-7643-7698-8>.
- [58] Harbarth K., Langer H. A factorization theorem for operator pencils // Integral Equations Operator Theory. — 1979. — Vol. 2, no. 3. — P. 344–364. — URL: <http://dx.doi.org/10.1007/BF01682674>.
- [59] Harte R. Spectral mapping theorems: a bluffer's guide. Springer Briefs in Mathematics. — Cham–Heidelberg–New York–Dordrecht–London : Springer, 2014. — xiv+120 p. — ISBN: 978-3-319-05647-0; 978-3-319-05648-7. — URL:

- <http://dx.doi.org/10.1007/978-3-319-05648-7>.
- [60] Harte R. E. Tensor products, multiplication operators and the spectral mapping theorem // Proc. Roy. Irish Acad., Sect. A. — 1973. — Vol. 73. — P. 285–302.
  - [61] Heinig G. The concepts of a bezoutiant and a resolvent for operator bundles // Funct. Anal. Appl. — 1977. — Vol. 11, no. 3. — P. 241–243.
  - [62] Heinig G., Rost K. Introduction to Bezoutians // Numerical methods for structured matrices and applications. — Basel : Birkhäuser Verlag, 2010. — Vol. 199 of Oper. Theory Adv. Appl. — P. 25–118.
  - [63] Heinz E. Beiträge zur Störungstheorie der Spektralzerlegung // Math. Ann. — 1951. — Vol. 123. — P. 415–438.
  - [64] Helemskii A. Y. Lectures and exercises on functional analysis. — Providence, RI : American Mathematical Society, 2006. — Vol. 233 of Translations of Mathematical Monographs. — xviii+468 p. — ISBN: 978-0-8218-4098-6; 0-8218-4098-3.
  - [65] Higham N. J. Functions of matrices: theory and computation. — Philadelphia, PA : Society for Industrial and Applied Mathematics (SIAM), 2008. — xx+425 p. — ISBN: 978-0-89871-646-7. — URL: <http://dx.doi.org/10.1137/1.9780898717778>.
  - [66] Higham N. J., Lin L. An improved Schur-Padé algorithm for fractional powers of a matrix and their Fréchet derivatives // SIAM J. Matrix Anal. Appl. — 2013. — Vol. 34, no. 3. — P. 1341–1360. — URL: <http://dx.doi.org/10.1137/130906118>.
  - [67] Hille E., Phillips R. S. Functional analysis and semi-groups. — Providence, RI : Amer. Math. Soc., 1957. — Vol. 31 of American Mathematical Society Colloquium Publications. — xiv+808 p.
  - [68] Ichinose T. On the spectra of tensor products of linear operators in Banach spaces. // J. Reine Angew. Math. — 1970. — Vol. 244. — P. 119–153.
  - [69] Ichinose T. Operational calculus for tensor products of linear operators in Banach spaces // Hokkaido Math. J. — 1975. — Vol. 4, no. 2. — P. 306–334.
  - [70] Ikramov K. D. Numerical solution of matrix equations. Orthogonal methods. — Moscow : Nauka, 1984. — 192 p. — (in Russian).
  - [71] Ikramov K. D. The nonsymmetric eigenvalue problem. — Moscow : Nauka, 1991. — 240 p. — (in Russian).
  - [72] Isaev G. A. Linear factorization of polynomial operator pencils // Mat. Zametki. — 1973. — Vol. 13. — P. 551–559.
  - [73] Jordan C. Calculus of finite differences. — Third edition. — New York : Chelsea Publishing Co., 1965. — xxi+655 p.
  - [74] Karplus R., Schwinger J. A note on saturation in microwave spectroscopy // Physical Review. — 1948. — Vol. 73, no. 9. — P. 1020–1026.
  - [75] Kato T. Perturbation theory for linear operators. Classics in Mathematics. — Berlin : Springer-Verlag, 1995. — xxii+619 p. — ISBN: 3-540-58661-X. — Reprint of the 1980 edition.
  - [76] Keldysh M. V. On the characteristic values and characteristic functions of certain classes of non-self-adjoint equations // Doklady Akad. Nauk SSSR (N.S.). — 1951. — Vol. 77, no. 1. — P. 11–14.
  - [77] Kenney C. S., Laub A. J. Condition estimates for matrix functions // SIAM J. Matrix Anal. Appl. — 1989. — Vol. 10, no. 2. — P. 191–209. — URL: <http://dx.doi.org/10.1137/0610014>.
  - [78] Kenney C. S., Laub A. J. A Schur-Fréchet algorithm for computing the logarithm and exponential of a matrix // SIAM J. Matrix Anal. Appl. — 1998. — Vol. 19, no. 3. — P. 640–663 (electronic). — URL:

- <http://dx.doi.org/10.1137/S0895479896300334>.
- [79] Kostyrykin V., Makarov K. A., Motovilov A. K. Existence and uniqueness of solutions to the operator Riccati equation. A geometric approach // Advances in differential equations and mathematical physics (Birmingham, AL, 2002). — Providence, RI : Amer. Math. Soc., 2003. — Vol. 327 of Contemp. Math. — P. 181–198. — URL: <http://dx.doi.org/10.1090/conm/327/05814>.
  - [80] Kostyrykin V., Makarov K. A., Motovilov A. K. On the existence of solutions to the operator Riccati equation and the  $\tan \Theta$  theorem // Integral Equations Operator Theory. — 2005. — Vol. 51, no. 1. — P. 121–140.
  - [81] Kostyuchenko A. G., Shkalikov A. A. Self-adjoint quadratic operator pencils and elliptic problems // Funct. Anal. Appl. — 1983. — Vol. 17, no. 2. — P. 109–128.
  - [82] Kreĭn M. G. Some new studies in the theory of perturbations of self-adjoint operators // First Math. Summer School, Part I. — Kiev : Naukova Dumka, 1964. — P. 103–187. — (Russian).
  - [83] Kreĭn M. G., Langer H. On some mathematical principles in the linear theory of damped oscillations of continua. I // Integral Equations Operator Theory. — 1978. — Vol. 1, no. 3. — P. 364–399. — URL: <http://dx.doi.org/10.1007/BF01682844>.
  - [84] Kreĭn M. G., Naĭmark M. A. The method of symmetric and Hermitian forms in the theory of the separation of the roots of algebraic equations // Linear and Multilinear Algebra. — 1981. — Vol. 10, no. 4. — P. 265–308. — URL: <http://dx.doi.org/10.1080/03081088108817420>.
  - [85] Kressner D. Bivariate matrix functions // Operators and Matrices. — 2014. — Vol. 8, no. 2. — P. 449–466.
  - [86] Kressner D., Tobler C. Krylov subspace methods for linear systems with tensor product structure // SIAM J. Matrix Anal. Appl. — 2010. — Vol. 31, no. 4. — P. 1688–1714.
  - [87] Krupnik I. N. Decomposition of a matrix pencil into linear factors // Mat. Zametki. — 1991. — Vol. 49, no. 2. — P. 95–101. — URL: <http://dx.doi.org/10.1007/BF01137549>.
  - [88] Kurbatov V. G., Kurbatova I. V. Krylov subspace methods of approximate solving of differential equations from the point of view of functional calculus // Eurasian Math. J. — 2012. — Vol. 3, no. 4. — P. 53–80.
  - [89] Kurbatov V. G., Kurbatova I. V. Extended tensor products and an operator-valued spectral mapping theorem // Izv. Math. — 2015. — Vol. 79, no. 4. — P. 710–739.
  - [90] Kurbatov V. G., Oreshina M. N. On approximate solution of second order linear differential equation // Vestnik Voronezhskogo gosudarstvennogo universiteta. Seriya: Fizika. Matematika [Proceedings of Voronezh State University. Series: Physics. Mathematics]. — 2003. — no. 2. — P. 173–188. — (in Russian). URL: [http://www.vestnik.vsu.ru/content/phymath/2003/02/toc\\_ru.asp](http://www.vestnik.vsu.ru/content/phymath/2003/02/toc_ru.asp).
  - [91] Kurbatov V. G., Oreshina M. N. Interconnect macromodelling and approximation of matrix exponent // Analog Integrated Circuits and Signal Processing. — 2004. — Vol. 40, no. 1. — P. 5–19.
  - [92] Kurbatova I. V. Some properties of linear relations // Vestnik fakulteta PMM [Proceedings of the faculty of Applied Mathematics]. — Voronezh : Voronezh State University, 2009. — Vol. 7. — P. 68–89. — (in Russian). URL: <https://www.researchgate.net/publication/269112381>.
  - [93] Kurbatova I. V. Functional calculus generated by a square pencil // Journal of Mathematical Sciences. — 2012. — Vol. 182, no. 5. — P. 646–655.

- [94] Lan N. T. On the operator equation  $AX - XB = C$  with unbounded operators  $A, B$ , and  $C$  // *Abstr. Appl. Anal.* — 2001. — Vol. 6, no. 6. — P. 317–328. — URL: <http://dx.doi.org/10.1155/S1085337501000665>.
- [95] Lumer G., Rosenblum M. Linear operator equations // *Proc. Amer. Math. Soc.* — 1959. — Vol. 10. — P. 32–41.
- [96] Markus A. S. Introduction to the spectral theory of polynomial operator pencils. — Providence, RI : American Mathematical Society, 1988. — Vol. 71 of *Translations of Mathematical Monographs*. — iv+250 p. — ISBN: 0-8218-4523-3.
- [97] Maroulas J., Psarrakos P. On factorization of matrix polynomials // *Linear Algebra Appl.* — 2000. — Vol. 304, no. 1-3. — P. 131–139. — URL: [http://dx.doi.org/10.1016/S0024-3795\(99\)00196-2](http://dx.doi.org/10.1016/S0024-3795(99)00196-2).
- [98] Mathias R. Evaluating the Fréchet derivative of the matrix exponential // *Numer. Math.* — 1992. — Vol. 63, no. 2. — P. 213–226. — URL: <http://dx.doi.org/10.1007/BF01385857>.
- [99] Mathias R. Approximation of matrix-valued functions // *SIAM J. Matrix Anal. Appl.* — 1993. — Vol. 14, no. 4. — P. 1061–1063. — URL: <http://dx.doi.org/10.1137/0614070>.
- [100] Najfeld I., Havel T. F. Derivatives of the matrix exponential and their computation // *Advances in Applied Mathematics*. — 1995. — Vol. 16. — P. 321–375.
- [101] Parlett B. A recurrence among the elements of functions of triangular matrices // *Linear Algebra and Appl.* — 1976. — Vol. 14, no. 2. — P. 117–121.
- [102] Perturbation theory for matrix equations / M. Konstantinov, Da-Wei Gu, V. Mehrmann, P. Petkov. — Amsterdam : North-Holland Publishing Co., 2003. — Vol. 9 of *Studies in Computational Mathematics*. — xii+429 p. — ISBN: 0-444-51315-9.
- [103] Pervozvanskii A. A. A Course in the Theory of Automatic Control. — Moscow : Nauka, 1986. — 615 p. — (in Russian).
- [104] Phóng V. Q. The operator equation  $AX - XB = C$  with unbounded operators  $A$  and  $B$  and related abstract Cauchy problems // *Math. Z.* — 1991. — Vol. 208, no. 4. — P. 567–588. — URL: <http://dx.doi.org/10.1007/BF02571546>.
- [105] Reed M., Simon B. Tensor products of closed operators on Banach spaces // *J. Functional Analysis*. — 1973. — Vol. 13. — P. 107–124.
- [106] Reed M., Simon B. *Methods of modern mathematical physics. I. Functional analysis.* — Second edition. — New York : Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], 1980. — xv+400 p. — ISBN: 0-12-585050-6.
- [107] Reed M. C., Simon B. A spectral mapping theorem for tensor products of unbounded operators // *Bull. Amer. Math. Soc.* — 1972. — Vol. 78. — P. 730–733.
- [108] Rinehart R. F. The derivative of a matrix function // *Proc. Amer. Math. Soc.* — 1956. — Vol. 7. — P. 2–5.
- [109] Rosenblum M. On the operator equation  $BX - XA = Q$  // *Duke Math. J.* — 1956. — Vol. 23. — P. 263–269.
- [110] Rudin W. *Functional analysis.* *International Series in Pure and Applied Mathematics.* — Second edition. — New York : McGraw-Hill, Inc., 1973. — xiii+397 p.
- [111] Schaefer H. H. *Topological vector spaces.* — The Macmillan Co., New York; Collier-Macmillan Ltd., London, 1966. — P. ix+294.
- [112] Schatten R. A. *A theory of cross-spaces.* — N. J. : Princeton Univ. Press, 1950. — vii+153 p.
- [113] Schechter M. On the spectra of operators on tensor products // *J. Functional Analysis*. — 1969. — Vol. 4. — P. 95–99.



- [114] Seshu S., Reed M. B. Linear graphs and electrical networks. Addison-Wesley Series in the Engineering Sciences. — Mass.-London : Addison-Wesley Publishing Co., Inc., 1961. — x+315 p.
- [115] Shabat B. V. Introduction to complex analysis. Part II. Functions of several variables. — Providence, RI : Amer. Math. Soc., 1992. — Vol. 110 of Translations of Mathematical Monographs. — x+371 p.
- [116] Shaw S.-Y., Lin S. C. On the equations  $Ax = q$  and  $SX - XT = Q$  // J. Funct. Anal. — 1988. — Vol. 77, no. 2. — P. 352–363. — URL: [http://dx.doi.org/10.1016/0022-1236\(88\)90092-4](http://dx.doi.org/10.1016/0022-1236(88)90092-4).
- [117] Shkalikov A. A., Pliev V. T. Compact perturbations of strongly damped operator pencils // Math. Notes. — 1989. — Vol. 45, no. 2. — P. 167–174.
- [118] Simon B. Uniform crossnorms // Pacific J. Math. — 1973. — Vol. 46. — P. 555–560.
- [119] Simoncini V. On the numerical solution of  $AX - XB = C$  // BIT. Numerical Mathematics. — 1996. — Vol. 36, no. 4. — P. 814–830. — URL: <http://dx.doi.org/10.1007/BF01733793>.
- [120] Computational methods for linear matrix equations : Rep. / Universit'a di Bologna ; Executor: V. Simoncini : 2013. — March. — 58 p.
- [121] Stickel E. On the Fréchet derivative of matrix functions // Linear Algebra Appl. — 1987. — Vol. 91. — P. 83–88.
- [122] Sylvester J. Sur l'équations en matrices  $px = xq$  // C. R. Acad. Sci. Paris. — 1884. — Vol. 99. — P. 67–71, 115–116.
- [123] Taylor A. E. Analysis in complex Banach spaces // Bull. Amer. Math. Soc. — 1943. — Vol. 49. — P. 652–669.
- [124] Taylor A. E. Spectral theory of closed distributive operators // Acta Math. — 1951. — Vol. 84. — P. 189–224.
- [125] Tisseur F., Meerbergen K. The quadratic eigenvalue problem // SIAM Rev. — 2001. — Vol. 43, no. 2. — P. 235–286. — URL: <http://dx.doi.org/10.1137/S0036144500381988>.
- [126] Van Loan C. The sensitivity of the matrix exponential // SIAM J. Numer. Anal. — 1977. — Vol. 14, no. 6. — P. 971–981.
- [127] Yosida K. Functional analysis. — Sixth edition. — Berlin–New York : Springer-Verlag, 1980. — Vol. 123 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. — P. xii+501. — ISBN: 3-540-10210-8.
- [128] Zayachkovskii V. S., Pankov A. A. Stability of factorization of polynomial operator pencils // Funct. Anal. Appl. — 1983. — Vol. 17, no. 2. — P. 140–142.
- [129] Zayachkovskii V. S., Pankov A. A. Weak stability of factorizations of operator pencils // Journal of Mathematical Sciences. — 1990. — Vol. 52, no. 6. — P. 3548–3550.
- [130] Zhu W., Xue J., Gao W. The sensitivity of the exponential of an essentially non-negative matrix // J. Comput. Math. — 2008. — Vol. 26, no. 2. — P. 250–258.

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